CS70: Countability and Uncountability

Alex Psomas

June 30, 2016

Warning!

Warning:

Warning!

Warning: I’m really loud!

Today.

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One idea, from around 130 years ago.

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At the heart of set theory.

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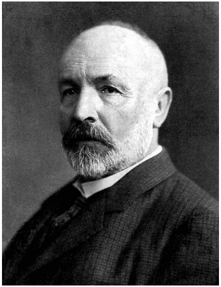

Georg Cantor

Life before Cantor

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How many elements in $\{1, 2, 4\}$?

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What is this infinity though?

The symbol you write after taking a limit....

Don't think about it....

Even Gauss: "I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction. "

Cantor’s questions

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Is $\mathbb{N} \setminus \{0\}$ smaller than $\mathbb{N}$?

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A hotel with infinite rooms.

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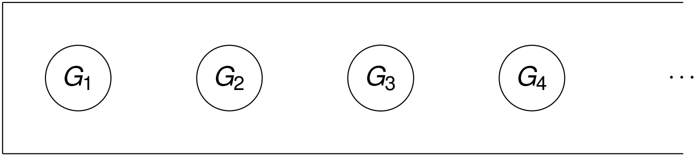

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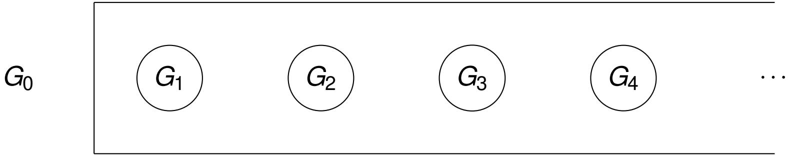

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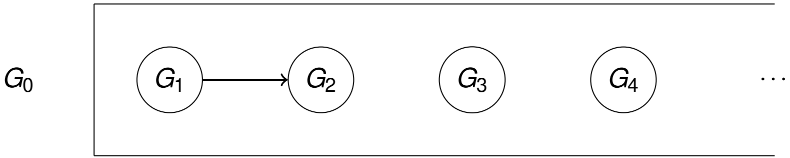

G_0 shows up. What do we do?

Move G_1 to room number 2.

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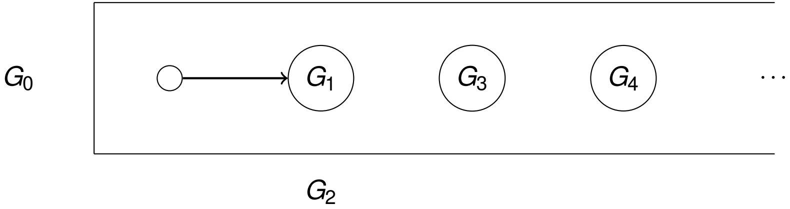

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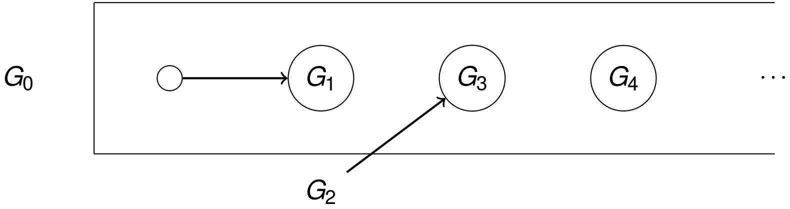

Move G_2 to room number 3.

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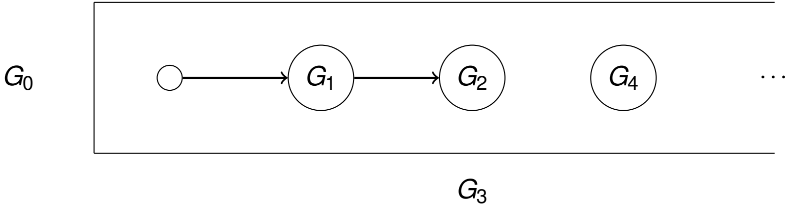

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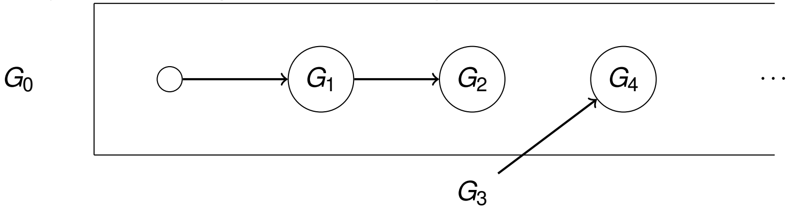

Move G_3 to room number 4.

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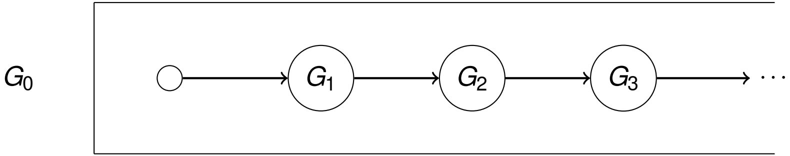

And so on.

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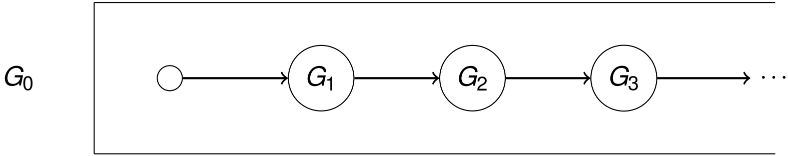

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Now G_0 can go to room number 1!!

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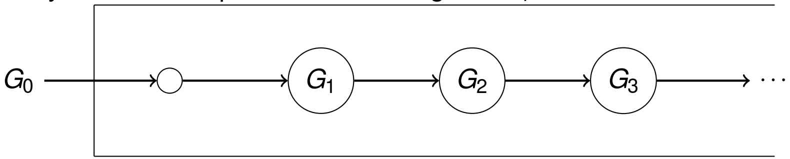

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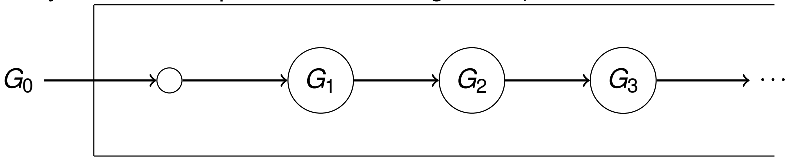

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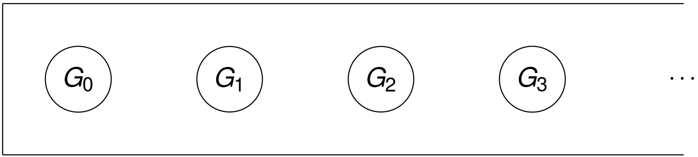

Moral of the story

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Is this a proof?

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Is this a proof? How would we show this formally???

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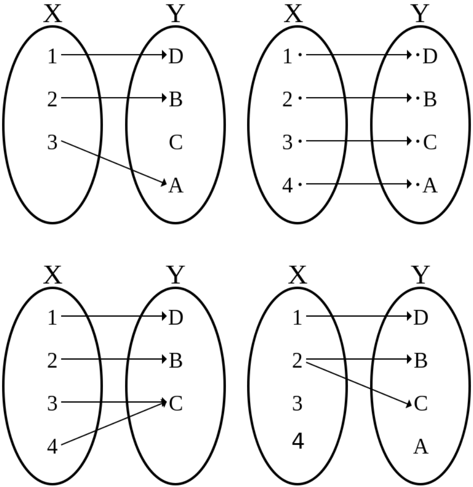
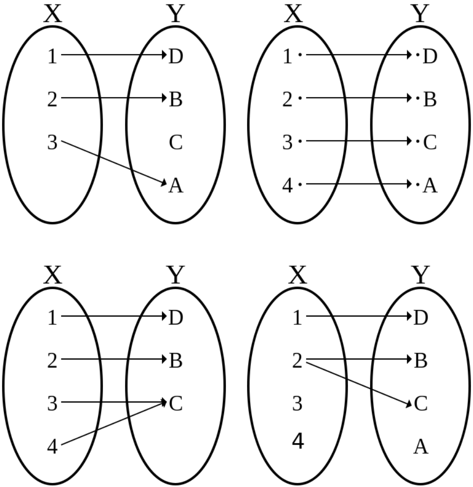
If the subset of $\mathbb{N}$ is infinite, S is **countably infinite**.

Bijections?



Bijections?

One to one.



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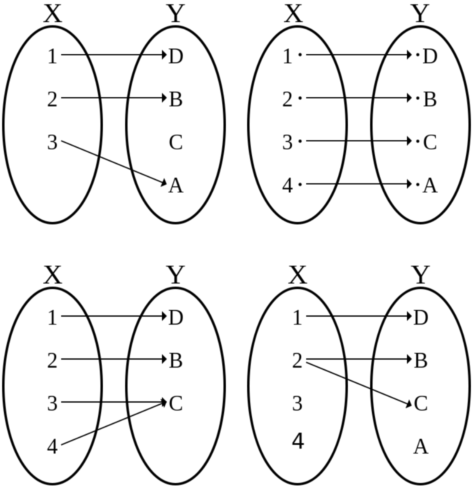
One to one. Bijection: one to one and onto.



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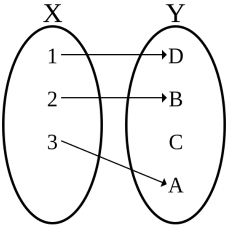

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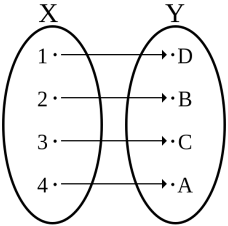

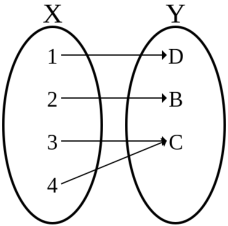

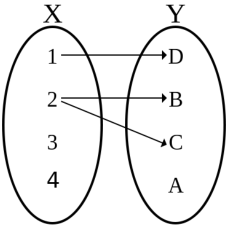

Onto.

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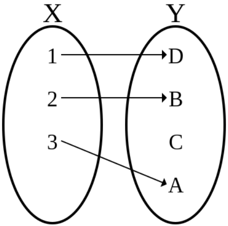

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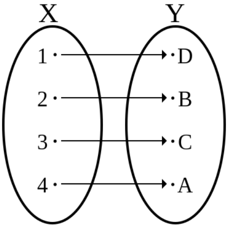

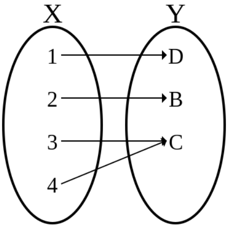

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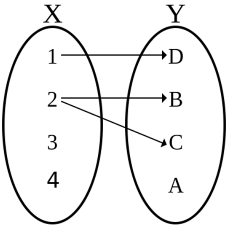

Not a function.

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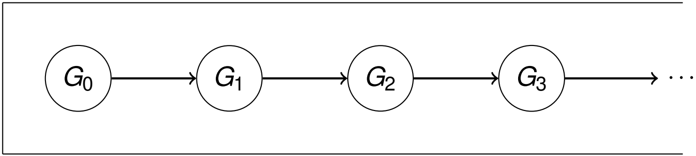
  For example the set $\{14, 54, 5332, 10^{12} + 4\}$ is countable. (It has 4 elements) Even numbers are countable. Prime numbers are countable. Multiples of 3 are countable.
- All countably infinite sets have the same cardinality as each other.

Back to Hilbert's hotel



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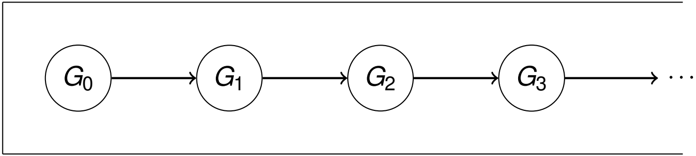
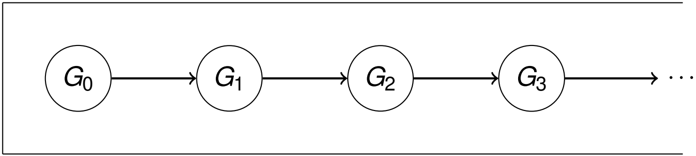


Where's the function?

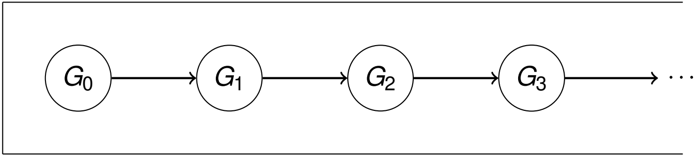
Back to Hilbert's hotel



Where's the function?

We want a bijection from:

Back to Hilbert's hotel



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We want a bijection from: $\mathbb{N} \setminus \{0\}$

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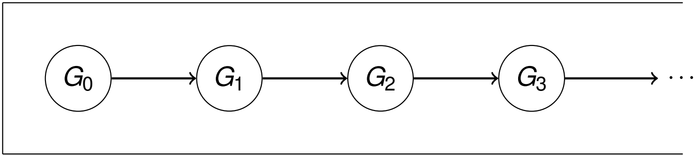

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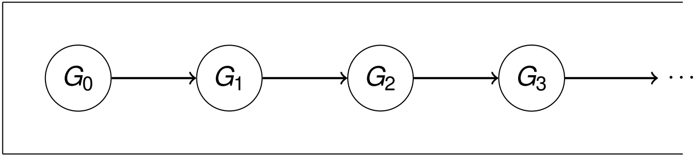

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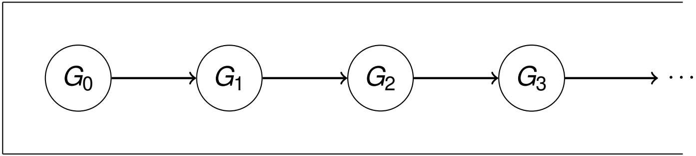

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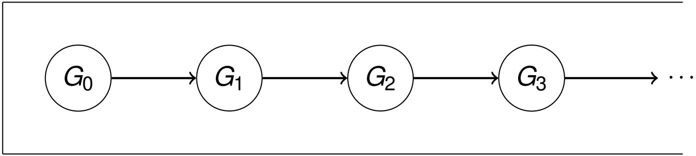

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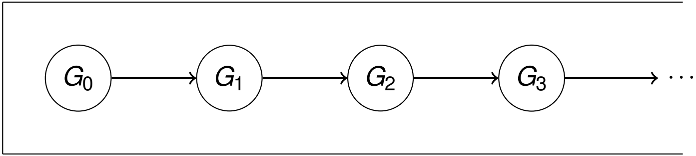

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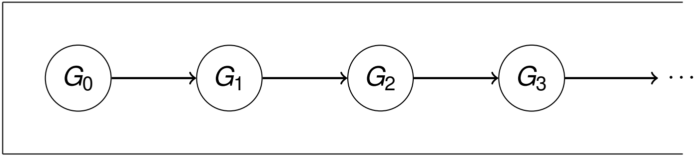

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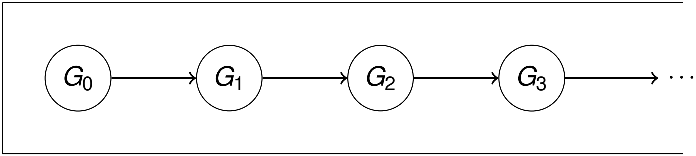

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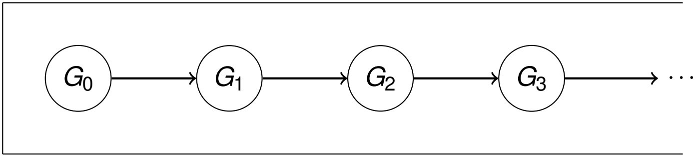

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Countably infinite (same cardinality as naturals)

- E even numbers.

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  Where $sign(z) = 1$ if $z > 0$ and $sign(z) = 0$ otherwise.

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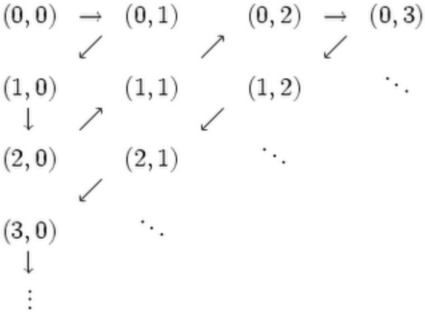

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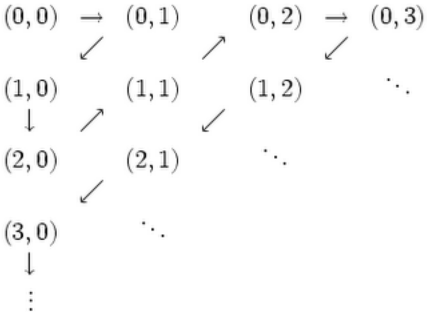

(a,b) at position $(a+b+1)(a+b)/2+b$ in this order.

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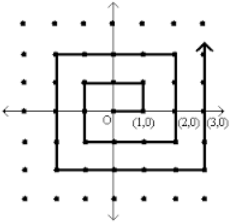

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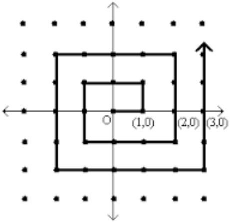

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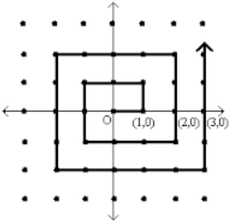

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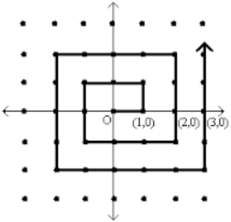

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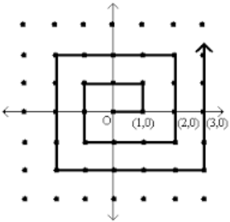

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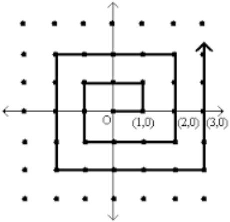

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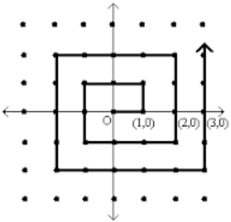

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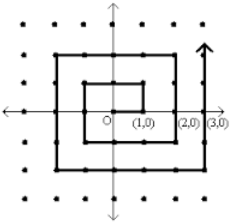

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Will eventually get to any pair.
Two different pairs cannot map to the same natural number/same position in the spiral.
Every natural has a "corresponding" pair.

Rationals

All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
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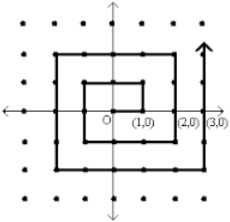

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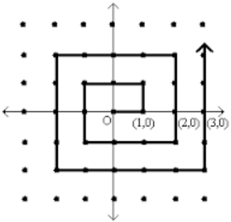

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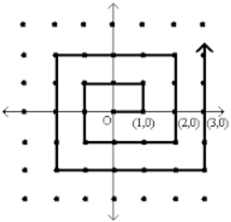

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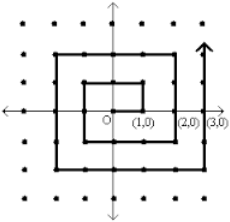

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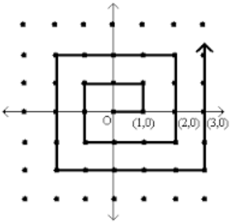

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Is the set of Reals countable?

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Subset $[0,1]$ is not countable!!

All reals?

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What about all reals?

All reals?

Subset $[0, 1]$ is not countable!!

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Any subset of a countable set is countable.

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If reals are countable then so is $[0, 1]$.

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1. Assume that a set S can be enumerated.

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Another diagonalization.

The set of all subsets of N.

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Example subsets of N: $\{0\}$,

Another diagonalization.

The set of all subsets of *N*.

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evens, odds, primes, multiples of 10

- Assume is countable.

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Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
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- Assume is countable.
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- *D* is a subset of *N*.

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- L does not contain all subsets of N.

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  Contradiction.

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  Contradiction.

Theorem: The set of all subsets of N is not countable.
(The set of all subsets of S, is the **powerset** of N.)

Another diagonalization.

$$
\begin{array}{l}
s_1 = 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots \\
s_2 = 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \ldots \\
s_3 = 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ \ldots \\
s_4 = 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ \ldots \\
s_5 = 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ \ldots \\
s_6 = 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ \ldots \\
s_7 = 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ \ldots \\
s_8 = 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ \ldots \\
s_9 = 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ \ldots \\
s_{10} = 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ \ldots \\
s_{11} = 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ \ldots \\
\vdots \qquad \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots
\end{array}
$$

$$
s = 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ \ldots
$$

Countable or uncountable??

- Binary strings?

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- Functions from $\mathbb{N}$ to $\mathbb{N}$?

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- Binary strings?
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You already know some of these.....

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You already know some of these..... Think about induction!

What happened with Cantor?

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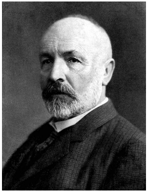

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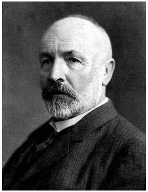

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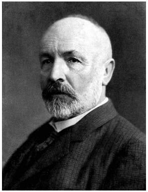

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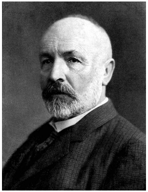

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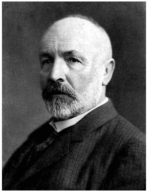

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- Countable

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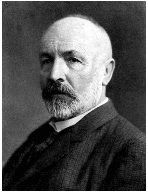

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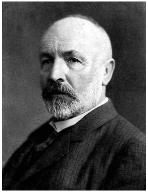

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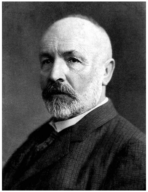

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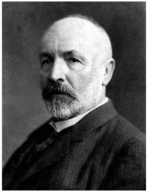

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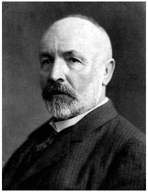

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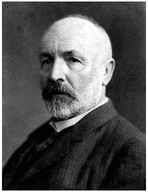

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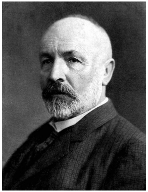

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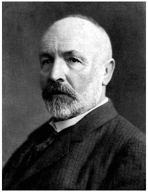

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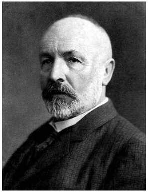

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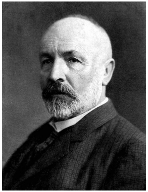

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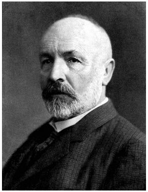

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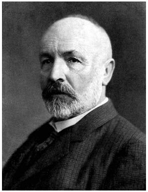

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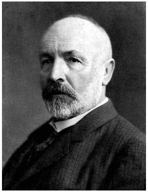

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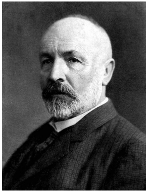

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Cantor's legacy

Gottlob Frege:

Cantor’s legacy

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Cantor's legacy

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Disaster!!

A bug

Bertrand Russell finds a bug!

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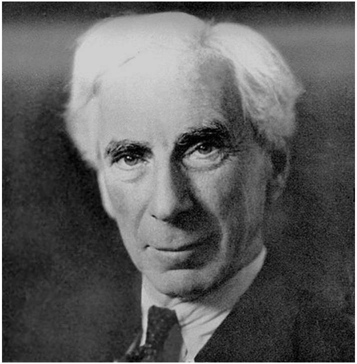

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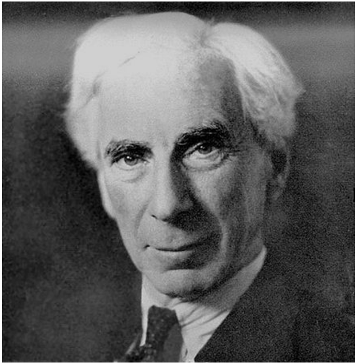

Frege's reaction:

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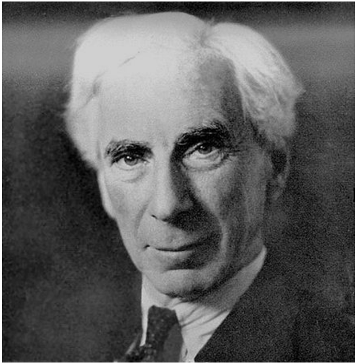

Frege's reaction: "Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion."

A poem

Zisimos Lorentzatos.
”Beware of systems grandiose, of mathematically strict causalities as you’re trying, stone by stone, to found the goldenwoven tower of the logical, castle and fort immune to contradiction. Designed in two volumes, the foundational laws of arithmetic, or Grundgesetze of der arithmetic in 1893, the first, 1903 the second. A life’s work. Hammer on chisel blows for years and years. So far, so good. But as Frege Gottlob was correcting, content, the printer’s proofs already of the second volume, one cursed logic paradox, one not admitting refutation, question by Russell Bertrand, forced, without delay, the great thinker of Mecklemburg to add a last paragraph to his system, show me a great thinker who would resist the truth, accepting the reversible disaster. His foundations in ruin, his logic flawed, his work wasted, and his two volumes imagine the colossal set back, odd load and ballast for the refuge cart.”

Russell's Paradox.

- ”This statement is false”

Russell’s Paradox.

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  Is the statement above true?
- A barber says ”I shave all and only those men who do not shave themselves.”

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  Who shaves the barber??

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- Self reference.........

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Let's think about the set of all sets that don't contain themselves.

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Does A contain itself?

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What type of object is a set that contain sets?

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Change Axioms!

Changing Axioms?

They did keep trying to put all of mathematics on a firm basis...

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Other people in this story:

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Other people in this story: Russell

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(We must know. We will know.) ...

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Until 1931.

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Concrete example:
Continuum hypothesis (see official notes if interested)

- Gödel

- Gödel ..starved himself out of fear of being poisoned..

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- Russell

- Gödel ..starved himself out of fear of being poisoned..
- Russell .. was fine...

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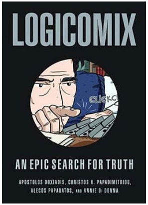
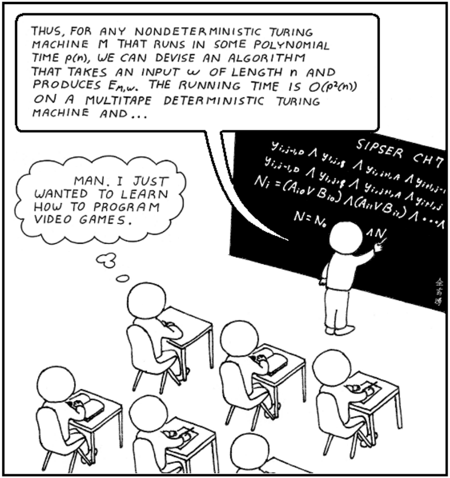
- See Logicomix by Doxiadis, Papadimitriou (my advisor!), Papadatos, Di Donna.



Next Topic: Undecidability.

- Undecidability. A happy ending?



Turing

Is it actually useful?

Turing: Write me a program checker!

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A program that checks that the compiler works!

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How about.. Check that the compiler terminates on a certain input.

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A program that checks that the compiler works!

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Theorem: There is no program HALT.

Halt does not exist.

Proof:

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Proof: Assume there is a program $HALT(\cdot, \cdot)$.

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P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Program P_1 halts on input $"P_1"$ and $"P_2"$, doesn't halt on input $"P_3"$, and so on...

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Wow, that was easy!

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We should be famous!

No computers for Turing!

In Turing’s time.

No computers for Turing!

In Turing's time.

No computers.

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In Turing's time.

No computers.

Concept of program as data wasn't really there.

Undecidable problems.

Does a program ever print “Hello World”?

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Find exit points and add statement: **Print** “Hello World.”

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Does this computer program have any security vulnerabilities?

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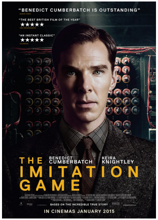

Turing: personal.

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2013. Granted Royal pardon.

Summary

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And mind boggling

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Computer Programs are an interesting thing.

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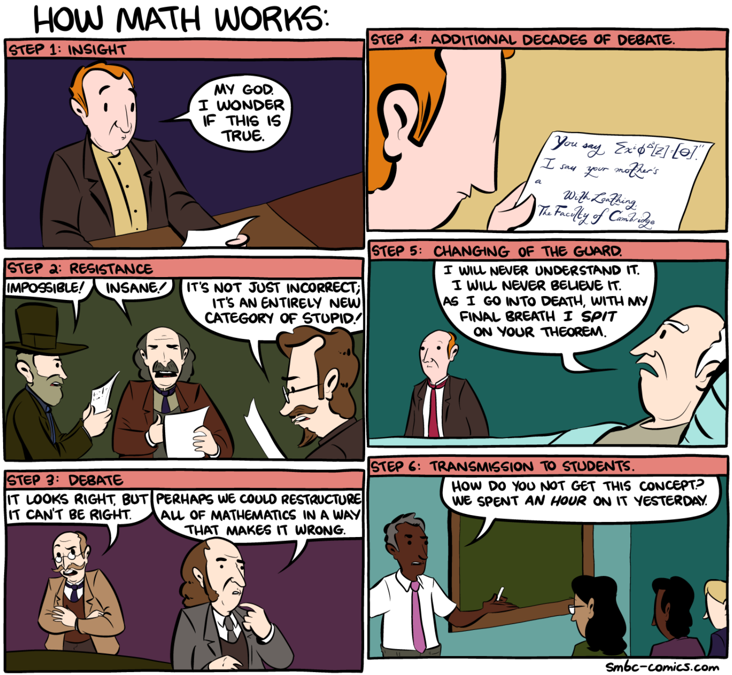
Programming is a super power.



HOW MATH WORKS:
STEP 1: INSIGHT
MY GOD. I WONDER IF THIS IS TRUE.
STEP 2: RESISTANCE
IMPOSSIBLE!
INSANE!
IT'S NOT JUST INCORRECT; IT'S AN ENTIRELY NEW CATEGORY OF STUPID!
STEP 3: DEBATE
IT LOOKS RIGHT, BUT IT CAN'T BE RIGHT.
PERHAPS WE COULD RESTRUCTURE ALL OF MATHEMATICS IN A WAY THAT MAKES IT WRONG.
STEP 4: ADDITIONAL DECADES OF DEBATE.
You say "$\Sigma x^i \phi^\Delta[z] = [\theta]$".
I say your mother's a
With Loathing,
The Faculty of Cambridge
STEP 5: CHANGING OF THE GUARD.
I WILL NEVER UNDERSTAND IT. I WILL NEVER BELIEVE IT. AS I GO INTO DEATH, WITH MY FINAL BREATH I *SPIT* ON YOUR THEOREM.
STEP 6: TRANSMISSION TO STUDENTS.
HOW DO YOU NOT GET THIS CONCEPT? WE SPENT AN *HOUR* ON IT YESTERDAY.
smbc-comics.com