CS70: Discrete Math and Probability

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June 29, 2016

Stable Marriage Problem

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- Small town with *n* boys and *n* girls.

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- Small town with n boys and n girls.
- Each girl has a ranked preference list of boys.

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- Each boy has a ranked preference list of girls.

Stable Marriage Problem

- Small town with n boys and n girls.
- Each girl has a ranked preference list of boys.
- Each boy has a ranked preference list of girls.

How should they be matched?

Count the ways..

- Maximize total satisfaction.

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.

The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

The best laid plans..

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- Jennifer and Brad
- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

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Brad prefers Angelina to Jennifer.

Angelina prefers Brad to BillyBob.

The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

Angelina prefers Brad to BillyBob.

Uh..oh.

So..

Produce a pairing where there is no running off!

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Produce a pairing where there is no running off!

Definition: A **pairing** is disjoint set of *n* boy-girl pairs.

So..

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Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$.

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b and g^* prefer each other to their partners in S

So..

Produce a pairing where there is no running off!

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Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$.

Definition: A **rogue couple** b, g^* for a pairing S:
b and g^* prefer each other to their partners in S

Example: Brad and Angelina are a rogue couple in S.

A stable pairing??

Given a set of preferences.

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Is there a stable pairing?

How does one find it?

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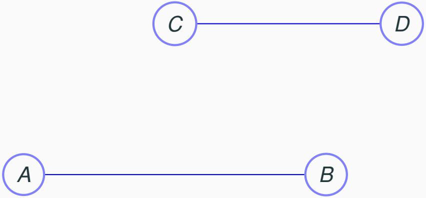
Given a set of preferences.

Is there a stable pairing?

How does one find it?

Consider a single gender version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



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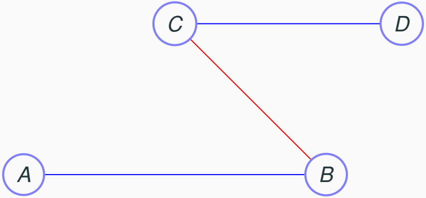

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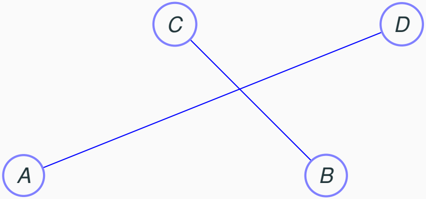

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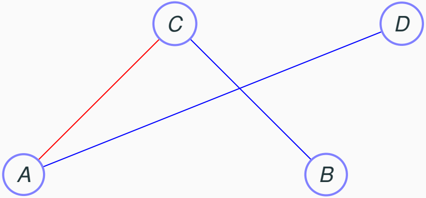

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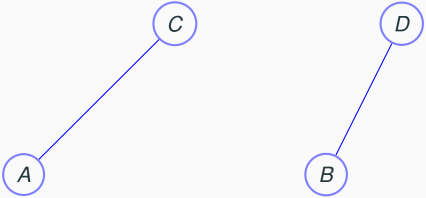

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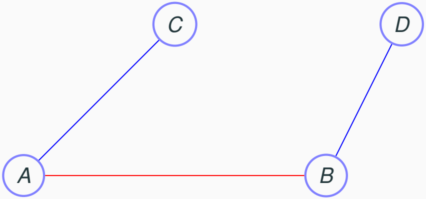

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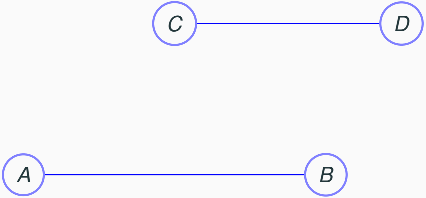

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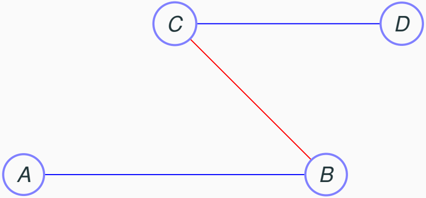

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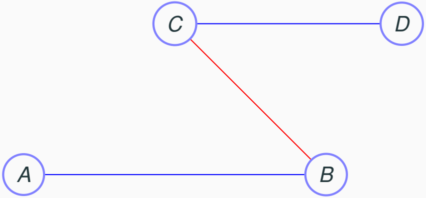

The Traditional Marriage Algorithm.

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Each Day:

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1. Each boy **proposes** to his favorite girl on his list.

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3. Rejected boy crosses rejecting girl off his list.

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Stop when each girl gets exactly one proposal.

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Does this terminate?

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Do boys or girls do “better”?

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Does this terminate?

...produce a pairing?

....a stable pairing?

Do boys or girls do “better”?

Example.

Boys			
A	1	2	3
B	1	2	3
C	2	1	3

Girls			
1	C	A	B
2	A	B	C
3	A	C	B

Example.

Boys			
A	1	2	3
B	1	2	3
C	2	1	3

Girls			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1					
2					
3					

Example.

Boys			
A	1	2	3
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Girls			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, B				
2	C				
3					

Example.

Boys			
A	1	2	3
B	~~1~~	2	3
C	2	1	3

Girls			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~				
2	C				
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Example.

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1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A			
2	C	B, C			
3					

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1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
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A	1	2	3
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Girls			
1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	A, C		
2	C	B, ~~C~~	B		
3					

Example.

Boys			
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1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~, C		
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Example.

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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~, C	C	
2	C	B, ~~C~~	B	A,B	
3					

Example.

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A	~~1~~	2	3
B	~~1~~	~~2~~	3
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1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~, C	C	
2	C	B, ~~C~~	B	A, ~~B~~	
3					

Example.

Boys			
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1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~, C	C	C
2	C	B, ~~C~~	B	A, ~~B~~	A
3					B

Example.

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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~, C	C	C
2	C	B, ~~C~~	B	A, ~~B~~	A
3					B

Termination.

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Every non-terminated day a boy **crossed** an item off the list.

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Total size of lists?

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Every non-terminated day a boy **crossed** an item off the list.

Total size of lists? *n* boys, *n* length list.

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Total size of lists? n boys, n length list. n^2

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Total size of lists? n boys, n length list. n^2

Terminates in at most $n^2 + 1$ steps!

It gets better every day for girls..

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Improvement Lemma: It just gets better for girls.

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If on day t a girl, g, has a boy b on a string,

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If on day t a girl, g, has a boy b on a string,
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Proof:
$P(k)$- - “boy on g's string is at least as good as b on day $t+k$”

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$P(0)$– true. Girl has b on string.

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On day $t+k+1$, boy b' comes back.
Girl can choose b',

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Girl can choose b', or do better with another boy, b''

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That is, $b \leq b'$ by induction hypothesis.

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Pairing when done.

Lemma: Every boy is matched at end.

Proof:
If not, a boy *b* must have been rejected *n* times.

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$\implies$ each girl has a boy on a string.

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Contradiction.

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b^* ——— g^*

b ——— g

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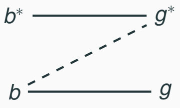

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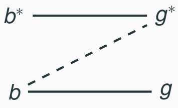

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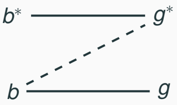

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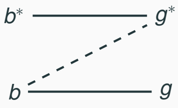

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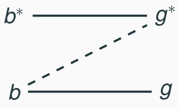

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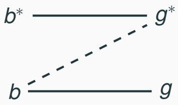

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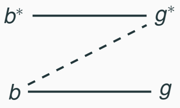

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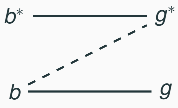

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Is the TMA better for boys?

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For boys?

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Theorem: TMA produces a boy-optimal pairing.

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Proof:

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Proof:
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TMA is optimal!

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Used Well-Ordering principle...Induction.

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Contradiction. $\square$

Notes: Not really induction.
Structural statement: Boy optimality

How about for girls?

Theorem: TMA produces girl-pessimal pairing.

T – pairing produced by TMA.

S – worse stable pairing for girl g.

In T, (g, b) is pair.

In S, (g, b^*) is pair.

g likes b^* less than she likes b.

T is boy optimal, so b likes g more than his partner in S.

(g, b) is Rogue couple for S

S is not stable.

Contradiction. □

Notes: Not really induction.
Structural statement: Boy optimality $\implies$ Girl pessimality.

Quick Questions.

How does one make it better for girls?

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SMA - stable marriage algorithm. One side proposes.

Quick Questions.

How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.
TMA - boys propose.

Quick Questions.

How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.
TMA - boys propose.
Girls could propose.

Quick Questions.

How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.
TMA - boys propose.
Girls could propose. $\implies$ optimal for girls.

Residency Matching..

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The method was used to match residents to hospitals.

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Hospital optimal....

Residency Matching..

The method was used to match residents to hospitals.

Hospital optimal....

..until 1990's...

Residency Matching..

The method was used to match residents to hospitals.

Hospital optimal....

..until 1990's...Resident optimal.

Summary

Summary

▸ Link

Summary

▸ Link

Tomorrow Alex starts on Infinity and Countability

Summary

▸ Link

Tomorrow Alex starts on Infinity and Countability

Thank you all!