# CS70: Discrete Math and Probability

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June 23, 2016

# More inductions!

Bunch of examples

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Bunch of examples
Good ones

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Bunch of examples
Good ones and bad ones

# More inductions

Suppose I start with 0 written on a piece of paper. Each time, I choose a digit written on the paper and erase it. If it was a 0, I replace it with 010. If it was a 1, I replace it with 1001. Prove that it's not possible for me to get two 1's in a row.

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Induct on number of operations I make.
Base case: After the first step, we get 010, which does not have two 1's in a row.

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Case 1: If we choose a 1, its neighbor must be 0's (based on ind hyp).

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By principle of induction, ... $\square$

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Case 2: $2^k < (n+1) < 2^{k+1}$, then we have $n+1 = 2^k + (n+1-2^k)$
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Done? Need to make sure $2^k$ is unique!
Since $n+1-2^k < 2^{k+1} - 2^k = 2^k$, all terms must have power less than $k$,

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By principle of induction, ... $\square$

# Strengthening: need to...

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

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$$= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.$$
$$\leq 2 + \frac{1}{(k+1)^2}$$

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Prove: $P(k+1)$

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Subtracting off a "quadratically decreasing" function every time.

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$$1 \leq \frac{k+1}{k} - \frac{1}{k+1}$$

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$1 \leq \frac{k+1}{k} - \frac{1}{k+1}$ Multiplied by $k+1$.

$1 \leq 1 + (\frac{1}{k} - \frac{1}{k+1})$

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$\implies S(k+1) \leq 2 - f(k+1)$.

Can you?

Subtracting off a ”quadratically decreasing” function every time.

Maybe a ”linearly decreasing” function to keep positive?

Try $f(k) = \frac{1}{k}$

$$\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2}?$$

$1 \leq \frac{k+1}{k} - \frac{1}{k+1}$ Multiplied by $k+1$.

$1 \leq 1 + (\frac{1}{k} - \frac{1}{k+1})$ Some math.

# Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

**Proof:**

Ind hyp: $P(k)$ — "$S_k \leq 2 - f(k)$"

Prove: $P(k+1)$ – "$S_{k+1} \leq 2 - f(k+1)$"

$$S(k+1) = S_k + \frac{1}{(k+1)^2}$$
$$\leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.}$$
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# Careful!

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As we will see, it is more subtle to catch errors in proofs of correct theorems!!

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By square both sides of the induction hypothesis we can get:

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Be careful.

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Definitions: model.

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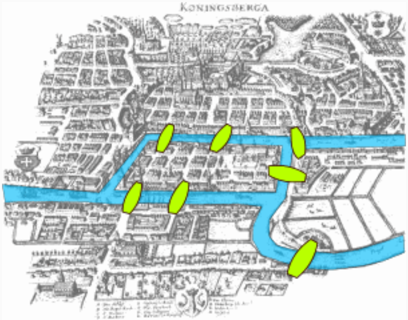
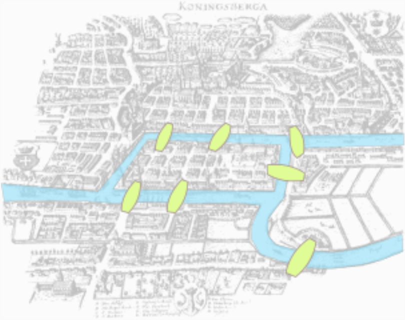
# Konigsberg bridges problem.

Can you make a tour visiting each bridge exactly once?

**Figure 1:** "Konigsberg bridges" by Bogdan Giuşcă - License.





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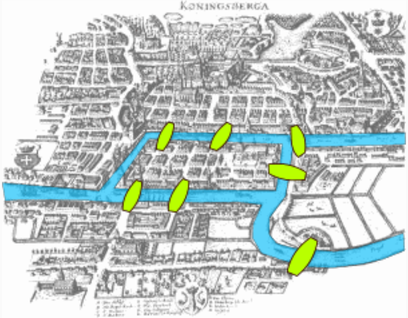


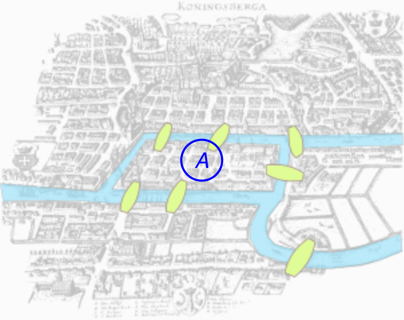

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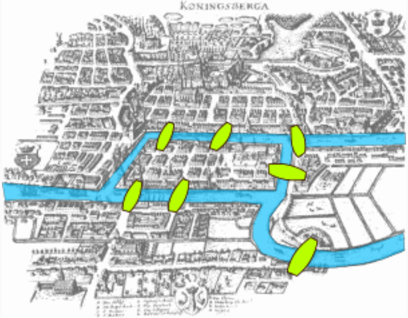


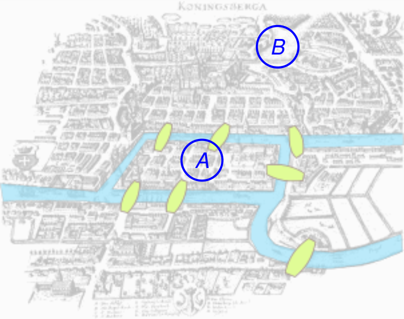

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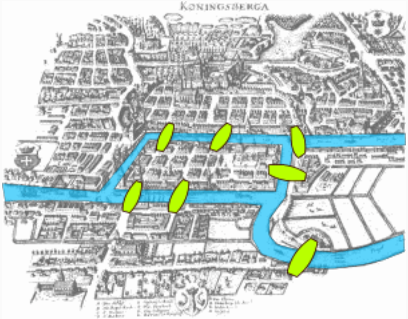


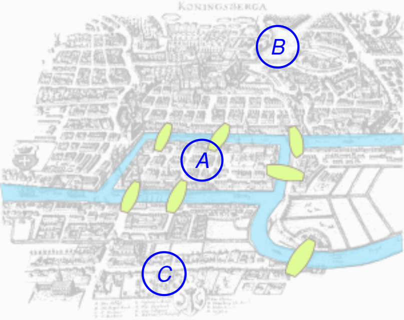

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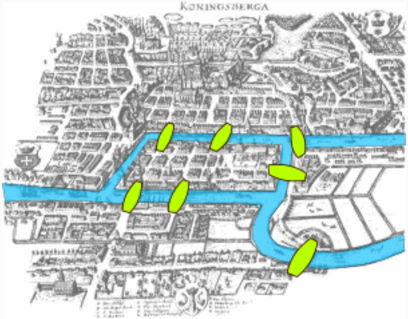


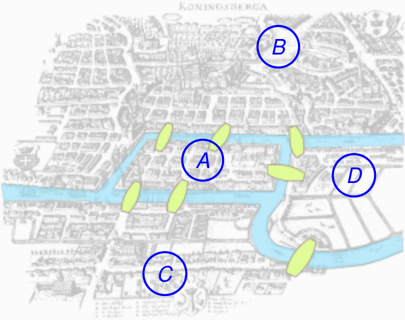

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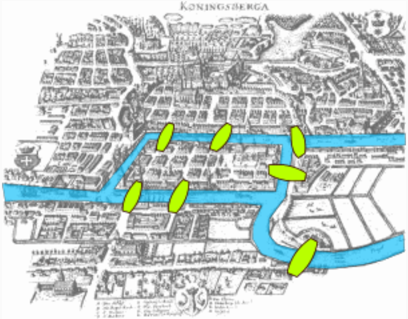

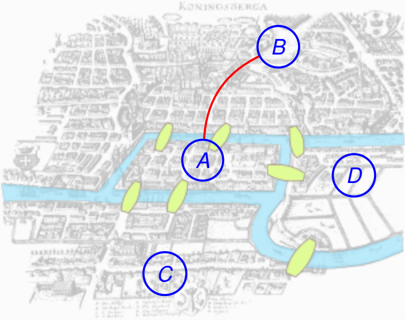

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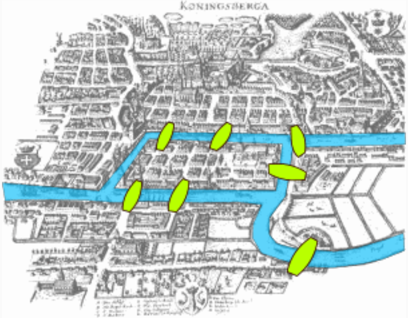


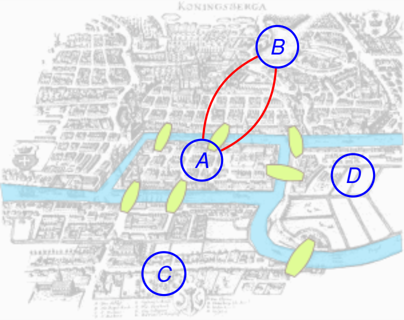

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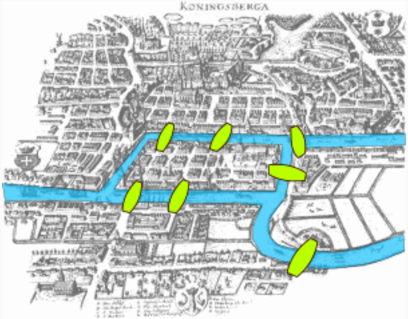


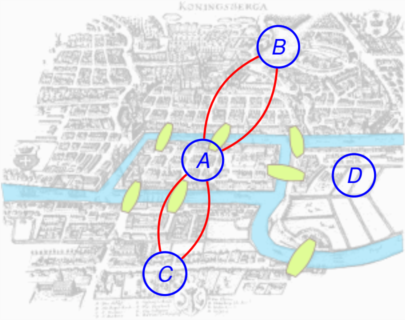

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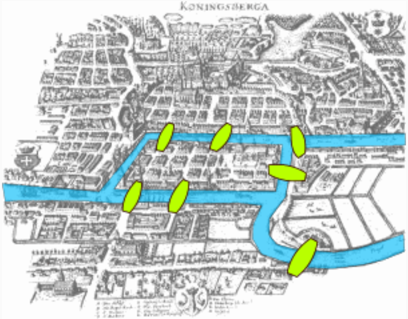


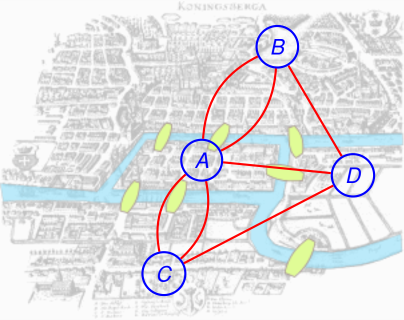

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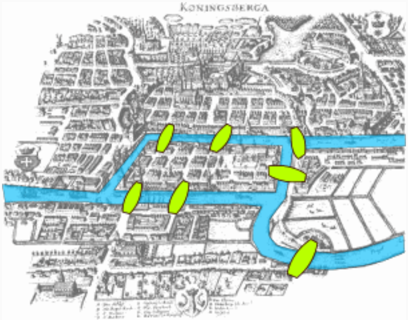
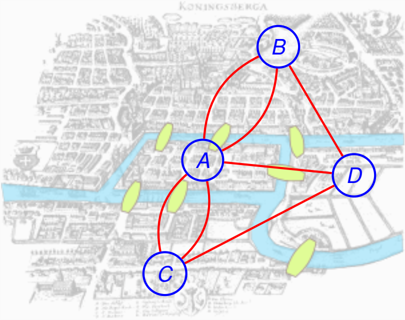


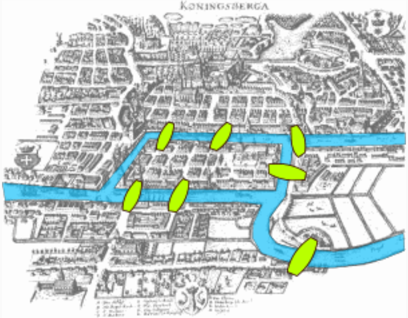



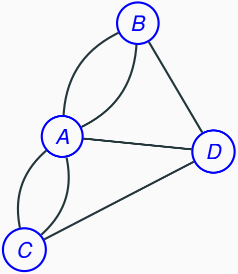

Can you draw a tour in the graph where you visit each edge once?

# Konigsberg bridges problem.

Can you make a tour visiting each bridge exactly once?

**Figure 1:** “Konigsberg bridges” by Bogdan Giuşcă - License.




Can you draw a tour in the graph where you visit each edge once? Yes?

# Konigsberg bridges problem.

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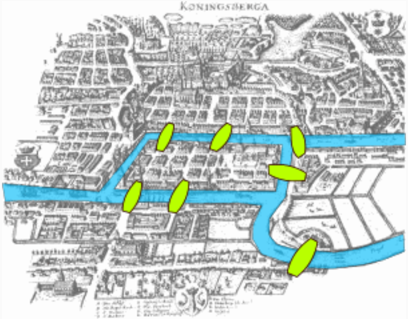


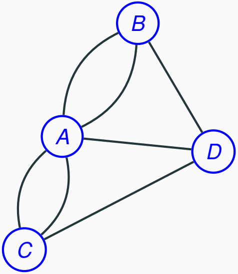


Can you draw a tour in the graph where you visit each edge once? Yes? No?

# Konigsberg bridges problem.

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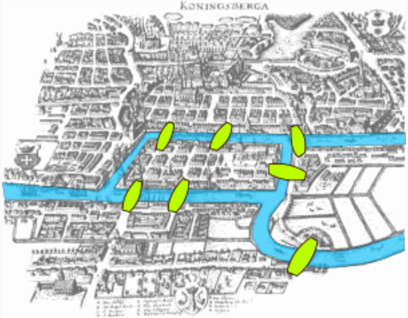
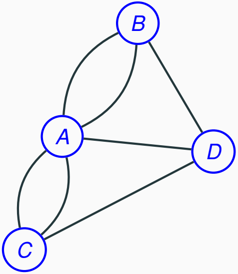
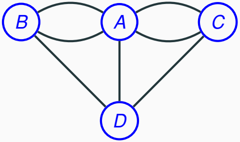



Can you draw a tour in the graph where you visit each edge once?   Yes?  No?
We will see!

# Graphs: formally.




Graph:

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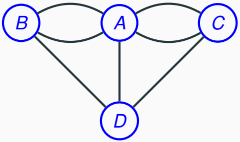
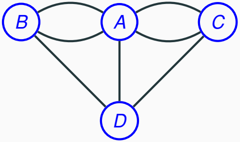



Graph: $G = (V, E)$.

# Graphs: formally.




Graph: $G = (V, E)$.

$V$ - set of vertices.

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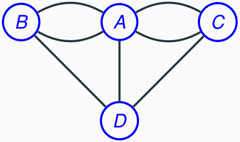


Graph: $G = (V, E)$.

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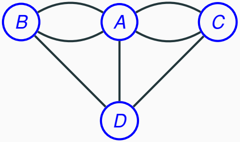
$\{A, B, C, D\}$

## Graphs: formally.

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$E \subseteq V \times V$ -

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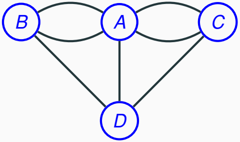


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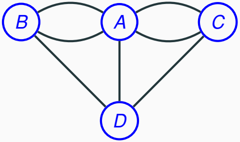


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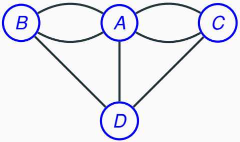


Graph: $G = (V, E)$.

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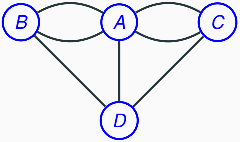


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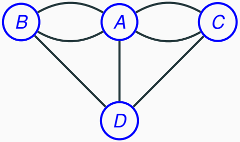


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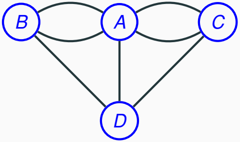


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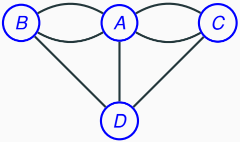
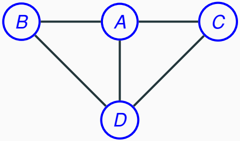
For CS 70, usually simple graphs.

# Graphs: formally.




Graph: $G = (V, E)$.
  $V$ - set of vertices.
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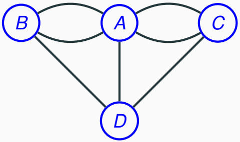
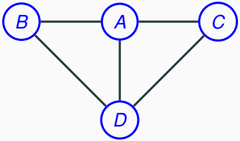
For CS 70, usually simple graphs.
  No parallel edges.

# Graphs: formally.




Graph: $G = (V, E)$.
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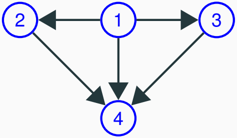
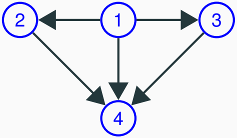
For CS 70, usually simple graphs.
No parallel edges.
Multigraph above.

# Directed Graphs



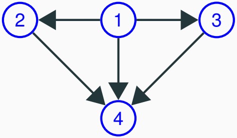

$G = (V, E)$.

# Directed Graphs




$G = (V, E)$.
$V$ - set of vertices.

# Directed Graphs

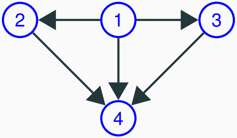



$G = (V, E)$.

$V$ - set of vertices.

$\{1, 2, 3, 4\}$

# Directed Graphs



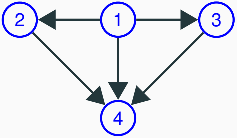

$G = (V, E)$.

$V$ - set of vertices.

$\{1, 2, 3, 4\}$

$E$ ordered pairs of vertices.

# Directed Graphs

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$\{(1, 2),$

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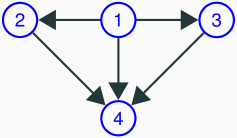


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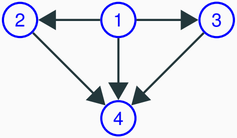


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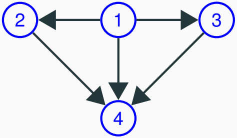


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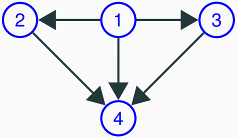


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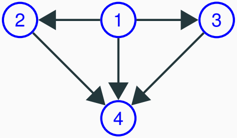
One way streets.

# Directed Graphs

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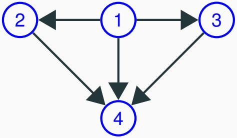
One way streets.
Tournament:

# Directed Graphs

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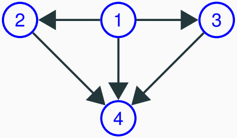
One way streets.
Tournament: 1 beats 2,

# Directed Graphs

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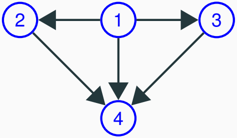
One way streets.
Tournament: 1 beats 2, ...
Precedence:

# Directed Graphs

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One way streets.
Tournament: 1 beats 2, ...
Precedence: 1 is before 2,

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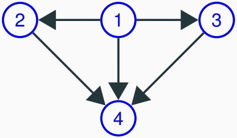


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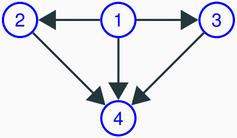


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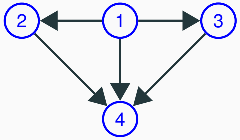
One way streets.
Tournament: 1 beats 2, ...
Precedence: 1 is before 2, ..

Social Network:

# Directed Graphs

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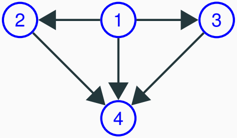
One way streets.
Tournament: 1 beats 2, ...
Precedence: 1 is before 2, ..

Social Network: Directed?

# Directed Graphs

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One way streets.
Tournament: 1 beats 2, ...
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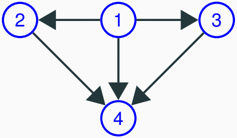
Social Network: Directed? Undirected?

# Directed Graphs

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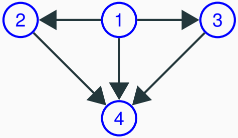
One way streets.
Tournament: 1 beats 2, ...
Precedence: 1 is before 2, ..

Social Network: Directed? Undirected?
Friends.

# Directed Graphs

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Tournament: 1 beats 2, ...
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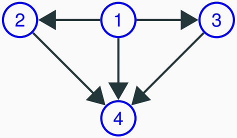
Social Network: Directed? Undirected?
Friends. Undirected.

# Directed Graphs

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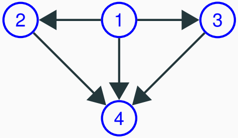
One way streets.
Tournament: 1 beats 2, ...
Precedence: 1 is before 2, ..

Social Network: Directed? Undirected?
Friends. Undirected.
Likes.

# Directed Graphs

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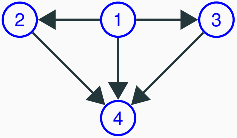
One way streets.
Tournament: 1 beats 2, ...
Precedence: 1 is before 2, ..

Social Network: Directed? Undirected?
Friends. Undirected.
Likes. Directed.

# Directed Graphs

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One way streets.
Tournament: 1 beats 2, ...
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Social Network: Directed? Undirected?
Friends. Undirected.
Likes. Directed.

# Graph Concepts and Definitions.

Graph: $G = (V, E)$

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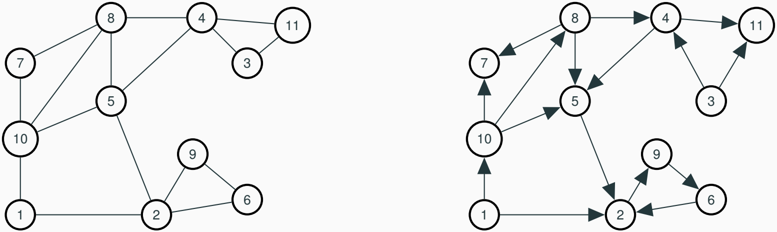
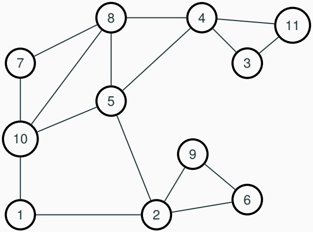


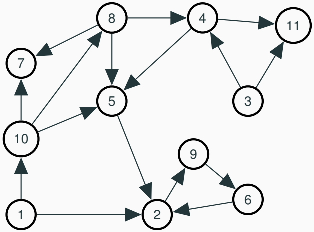

Neighbors of 10?

# Graph Concepts and Definitions.

Graph: $G = (V, E)$

neighbors, adjacent, degree, incident, in-degree, out-degree




Neighbors of 10? 1,

# Graph Concepts and Definitions.

Graph: $G = (V, E)$
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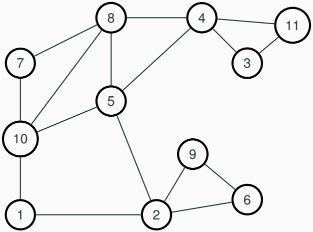


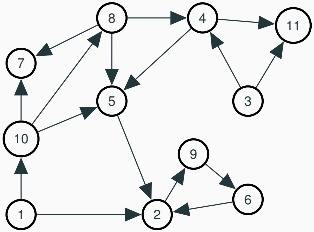


Neighbors of 10? 1,5,

# Graph Concepts and Definitions.

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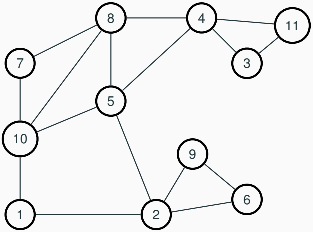


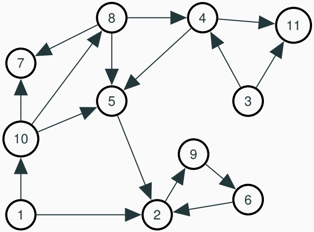


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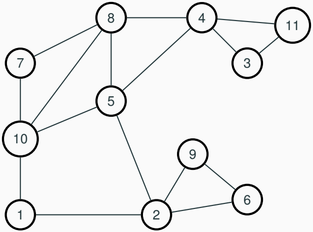


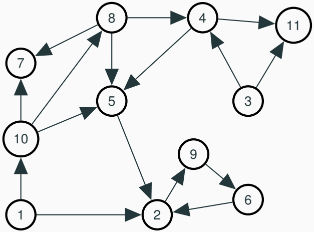


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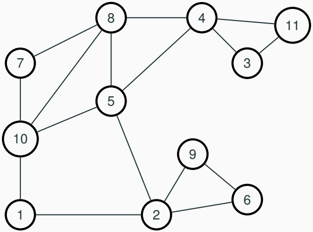


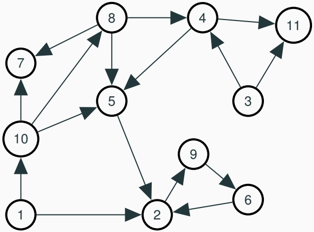


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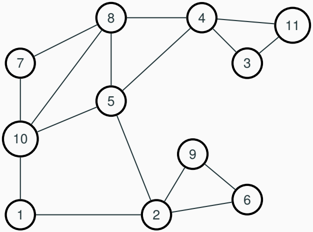


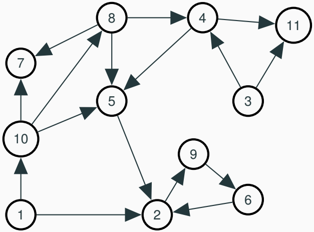


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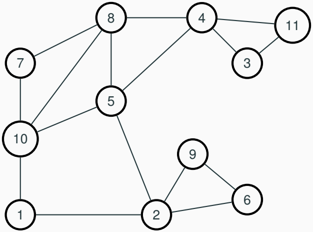


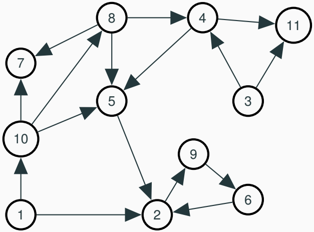


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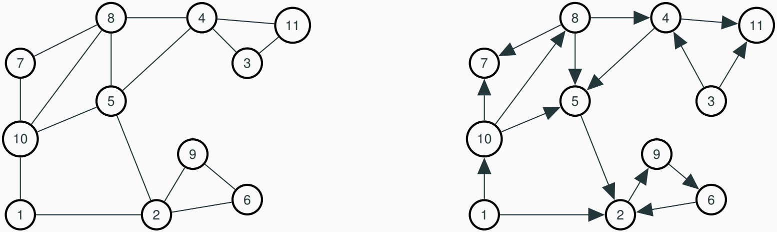


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Degree of vertex 1?

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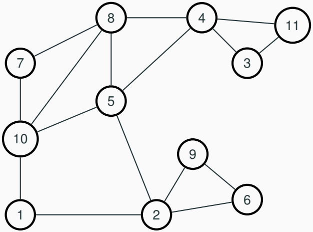


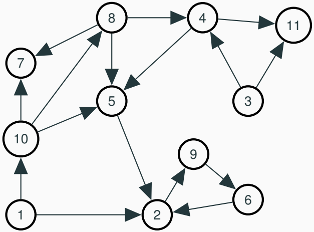


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Degree of vertex 1? 2

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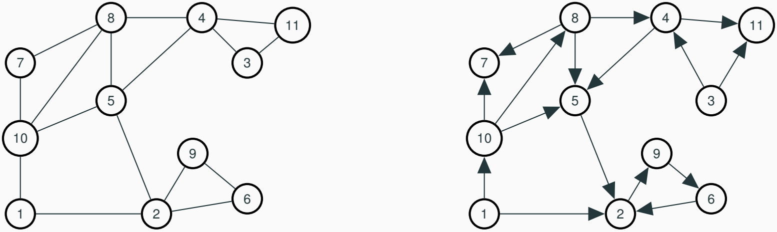


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Degree of vertex $u$ is number of incident edges.

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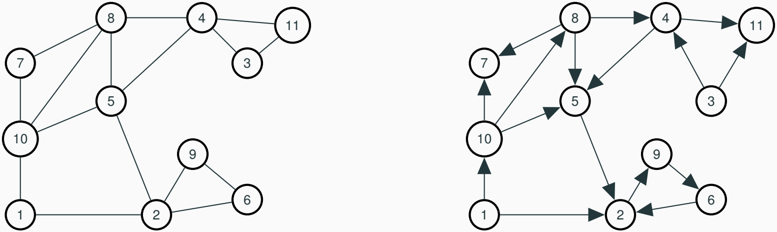


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Equals number of neighbors in simple graph.

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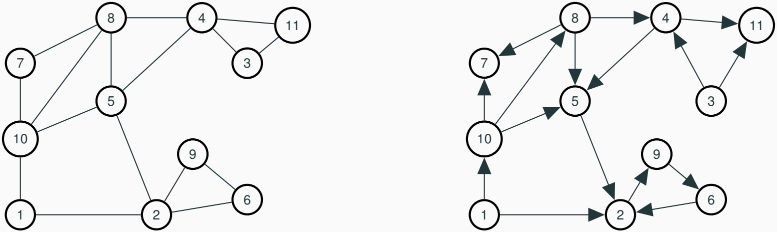


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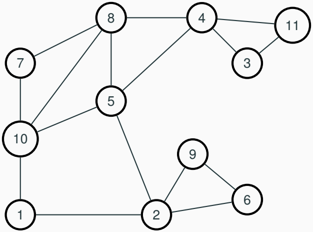


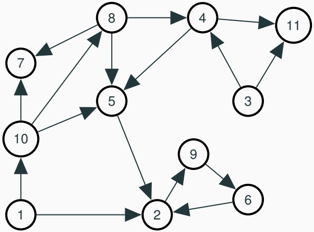


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Directed graph?

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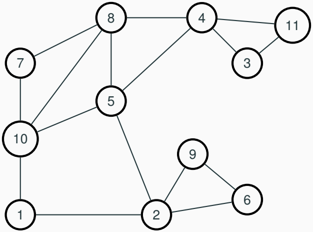


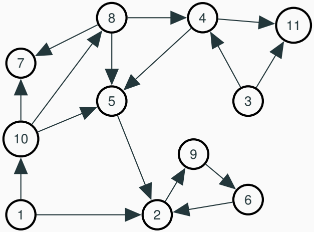


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In-degree of 10?

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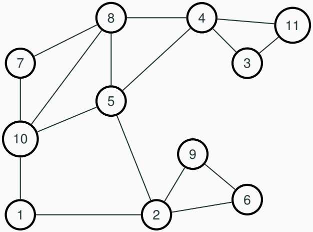


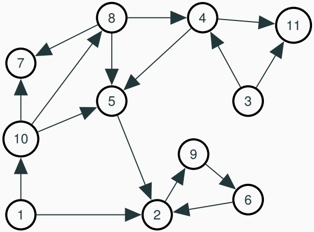


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Directed graph?
In-degree of 10? 1

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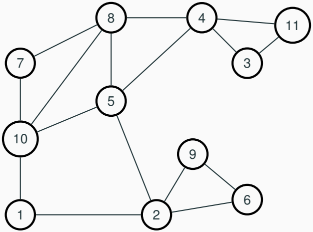


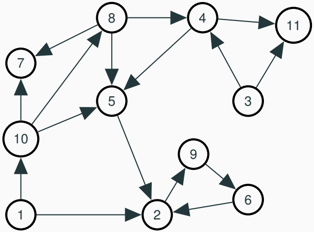


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In-degree of 10? 1    Out-degree of 10?

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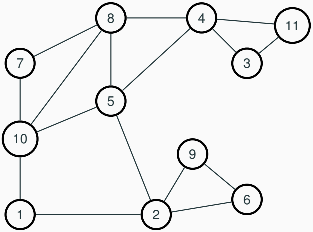


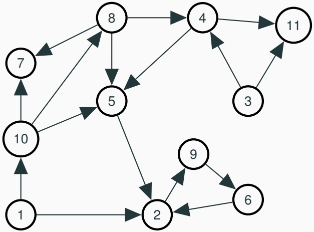


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Equals number of neighbors in simple graph.

Directed graph?
In-degree of 10? 1    Out-degree of 10? 3

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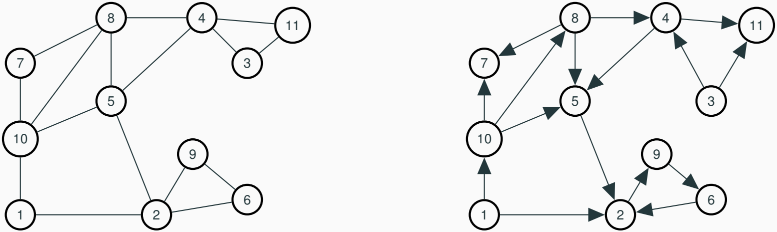


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The sum of the vertex degrees is equal to

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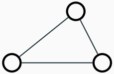

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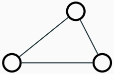

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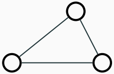

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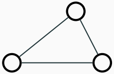

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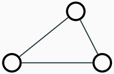

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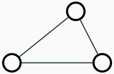

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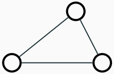

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Could it always be...$2|E|$?

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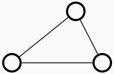

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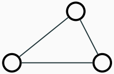

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How many incidences does each edge contribute? 2.

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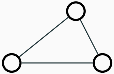

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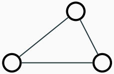

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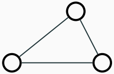

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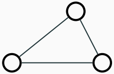

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Could it always be...$2|E|$?

How many incidences does each edge contribute? 2.
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What is degree $v$? incidences contributed to $v$!
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# Quick Proof.

The sum of the vertex degrees is equal to

(A) the total number of vertices, $|V|$.
(B) the total number of edges, $|E|$.
(C) What?

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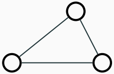

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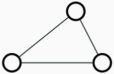

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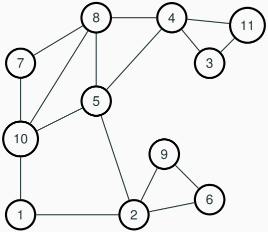
**Thm:** Sum of vertex degress is $2|E|$.

# Paths, walks, cycles, tour.




A path in a graph is a sequence of edges.

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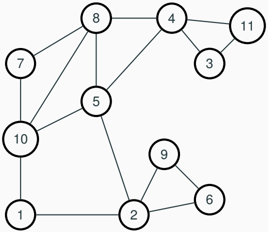


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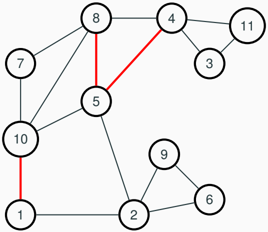


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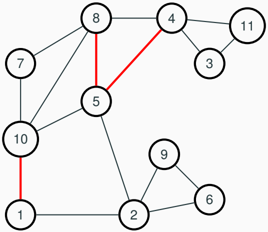


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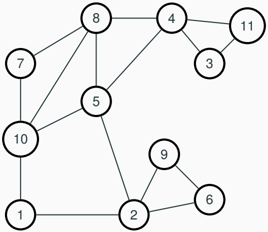


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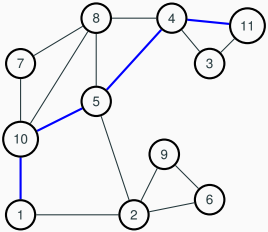


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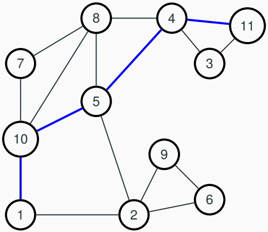


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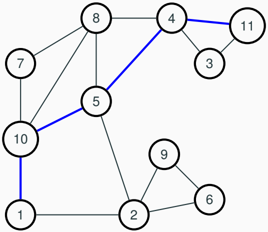


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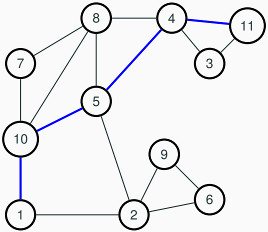


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Quick Check!

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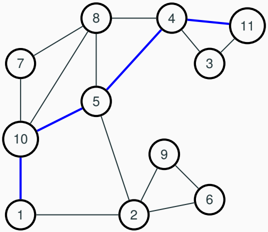


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Quick Check! Length of path?

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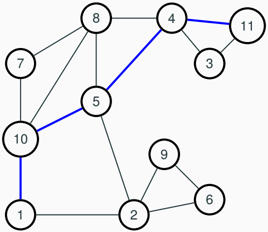


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Quick Check! Length of path? $k$ vertices

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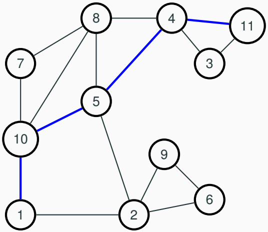


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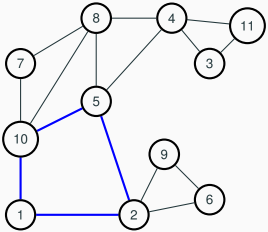


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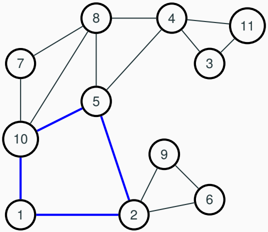


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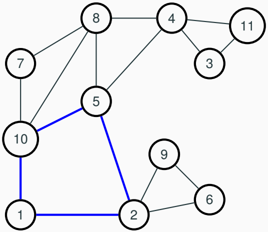


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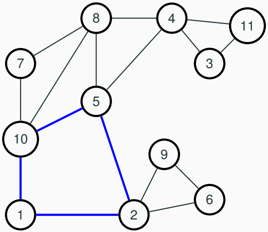


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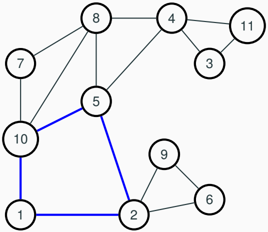


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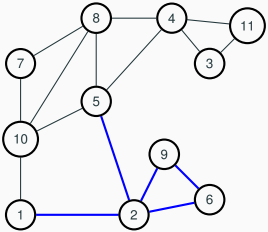


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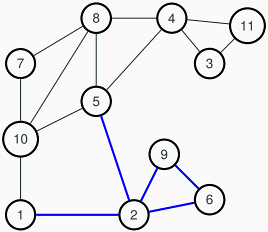


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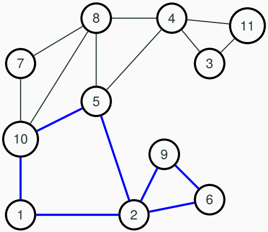


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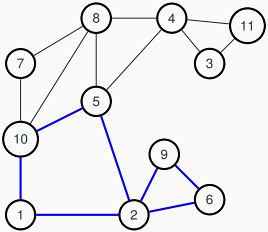


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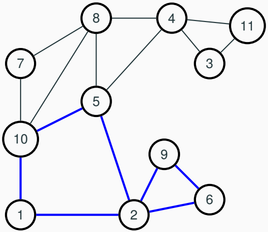


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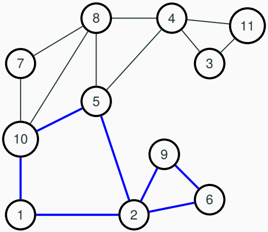


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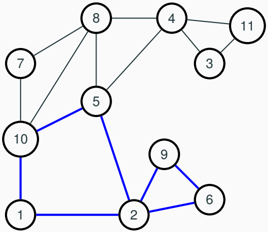


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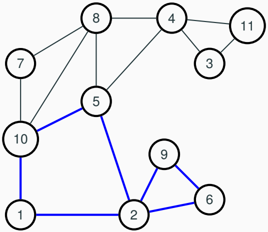


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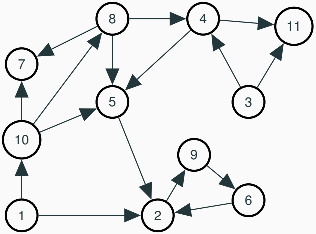

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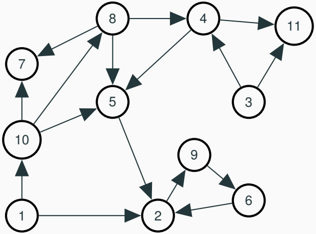


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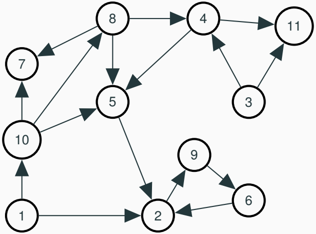


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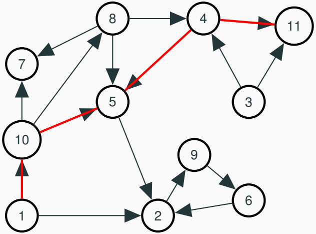


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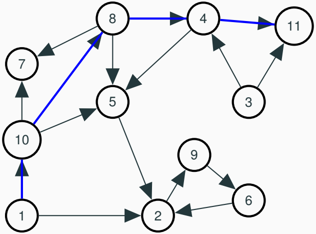


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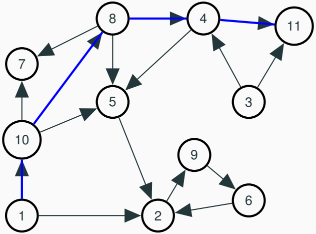


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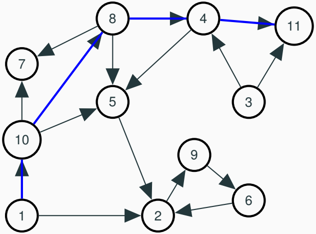


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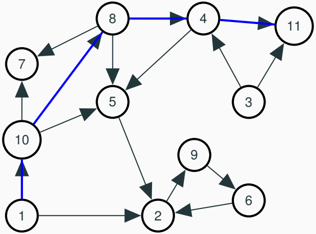


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Paths, walks, cycles,

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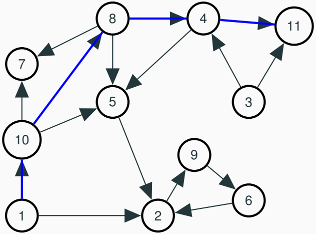


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Paths, walks, cycles, tours

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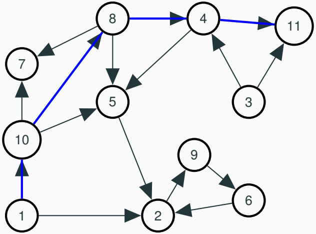


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Paths, walks, cycles, tours ... are analagous to undirected now.

# Thank you!

Congrats on surviving the first week!

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Have a good weekend!

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Don’t forget your homework, homework party tonight.