CS70: Discrete Math and Probability

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More inductions!

Bunch of examples
Good ones and bad ones

More inductions

Suppose I start with 0 written on a piece of paper. Each time, I choose a digit written on the paper and erase it. If it was a 0, I replace it with 010. If it was a 1, I replace it with 1001. Prove that it's not possible for me to get two 1's in a row.

Induct on number of operations I make.
Base case: After the first step, we get 010, which does not have two 1's in a row.

Ind hyp: Assume after n steps, we do not have two 1's in a row.

Ind steps: At the $(n+1)_{th}$ step,

Case 1: If we choose a 1, its neighbor must be 0's (based on ind hyp). Therefore after the change we will have $\dots$010010$\dots$, it cannot create two 1's in a row.

Case 2: If we choose a 0, and change it to 010, then its previous neighbors will still have 0 as their neighbors.Thus after the change we still do not have two 1's.

Therefore after the $(n+1)_{th}$ step there are still not two 1's in a row.
By principle of induction, ... □

More power, more sum!

Theorem: Every positive integer n can be written as a sum of distinct powers of 2.

Discuss with your neighbors
Vanilla induction? Strong induction? Strengthen ind hyp?

Base case: For $n = 1 = 2^0$, $p(1)$ is true.
Induction hypothesis:

Assume all integers between 1 and n can be written as sums of distinct powers of 2.

Induction steps:Need to show that $n+1$ can be written as a sum of distinct powers of 2.
We can find a k such that $2^k \le (n+1) < 2^{k+1}$

Case 1: $n+1 = 2^k$, we are done

Case 2: $2^k < (n+1) < 2^{k+1}$, then we have $n+1 = 2^k + (n+1-2^k)$
Base on induction hypothesis, $n+1-2^k$ can be written as a sum of distinct powers of 2.
Done? Need to make sure 2^k is unique!
Since $n+1-2^k < 2^{k+1} - 2^k = 2^k$, all terms must have power less than k, thus 2^k must be unique in this sum for $n+1$.

By principle of induction, ... □

Strengthening: need to...

Theorem: For all $n \ge 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \le 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Base: $P(1)$. $1 \le 2$.
Ind Step: $\sum_{i=1}^{k} \frac{1}{i^2} \le 2$.

$\sum_{i=1}^{k+1} \frac{1}{i^2}$
$= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}$.
$\le 2 + \frac{1}{(k+1)^2}$

Uh oh?

Hmmm... It better be that any sum is *strictly less than* 2.

How much less? At least by $\frac{1}{(k+1)^2}$ for S_k.
"$S_k \le 2 - \frac{1}{(k+1)^2}$" $\implies$ "$S_{k+1} \le 2$"
Induction step works! No! Not the same statement!!!!
Need to prove "$S_{k+1} \le 2 - \frac{1}{(k+2)^2}$".

Oooops.....

Strengthening: how?

Theorem: For all $n \ge 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \le 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)
Proof:
Ind hyp: $P(k)$ — "$S_k \le 2 - f(k)$"
Prove: $P(k+1)$ – "$S_{k+1} \le 2 - f(k+1)$"

$S(k+1) = S_k + \frac{1}{(k+1)^2}$
$\le 2 - f(k) + \frac{1}{(k+1)^2}$ By ind. hyp.
need to show: $\le 2 - f(k+1)$

Choose $f(k+1) \le f(k) - \frac{1}{(k+1)^2}$.
$\implies S(k+1) \le 2 - f(k+1)$.

Can you?
Subtracting off a "quadratically decreasing" function every time.
Maybe a "linearly decreasing" function to keep positive?
Try $f(k) = \frac{1}{k}$

$\frac{1}{k+1} \le \frac{1}{k} - \frac{1}{(k+1)^2}$?

$1 \le \frac{k+1}{k} - \frac{1}{k+1}$ Multiplied by $k+1$.
$1 \le 1 + (\frac{1}{k} - \frac{1}{k+1})$ Some math. So yes!

Theorem: For all $n \ge 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \le 2 - \frac{1}{n}$.

Careful!

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.
New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?

First k have same color by $P(k)$. $1,2,3,\ldots,k,k+1$ $1,2$
Second k have same color by $P(k)$. $1,2,3,\ldots,k,k+1$ $1,2$
A horse in the middle in common! $1,2,3,\ldots,k,k+1$ $1,2$
All k must have the same color. $1,2,3,\ldots,k,k+1$
No horse in common!

How about $P(1) \implies P(2)$?

Fix base case.
...Still doesn't work!!
(There are two horses is $\not\equiv$ For all two horses!!!)

Of course it doesn't work.

As we will see, it is more subtle to catch errors in proofs of correct theorems!!

Good? Bad? –1

Use induction to prove the follow equality:

$$n = \sqrt{1+(n-1)\sqrt{1+n\sqrt{1+(n+1)\sqrt{1+(n+2)\ldots}}}}$$

for all positive integers n.

Proof by induction:
Base case: for $n = 1$, $1 = \sqrt{1+0} = 1$, equality holds.
Induction hypothesis: Assume this equality holds for $n = k$, i.e.

$$k = \sqrt{1+(k-1)\sqrt{1+k\sqrt{1+(k+1)\sqrt{1+(k+2)\ldots}}}}$$

Good? Bad? –2

Ind hyp:

$$k = \sqrt{1+(k-1)\sqrt{1+k\sqrt{1+(k+1)\sqrt{1+(k+2)\ldots}}}}$$

Induction step:Need to show it holds for $n = k+1$.
By square both sides of the induction hypothesis we can get:

$$k^2 = 1+(k-1)\sqrt{1+k\sqrt{1+(k+1)\sqrt{1+(k+2)\ldots}}}$$

Easy to get

$$\frac{k^2-1}{k-1} = k+1 = \sqrt{1+k\sqrt{1+(k+1)\sqrt{1+(k+2)\ldots}}}$$

Therefore it holds for $n = k+1$, by principle of induction, ... □Good or bad?

Good? Bad? –3

Bad proof! We need $k \neq 1$ to divide both sides by $k-1$

Or in other words, $p(1)$ does not imply $p(2)$

Be careful.

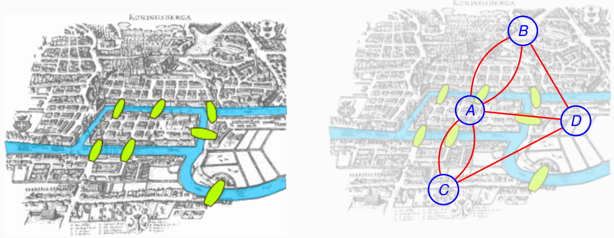
Note 4: Graph theory

Graphs!
Definitions: model.

Konigsberg bridges problem.

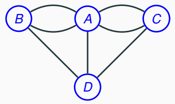
Can you make a tour visiting each bridge exactly once?

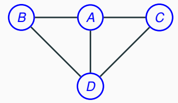
Figure 1: "Konigsberg bridges" by Bogdan Giuşcă - License.



Can you draw a tour in the graph where you visit each edge once? Yes? No?
We will see!

Graphs: formally.

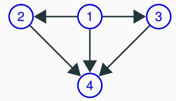




Graph: $G = (V, E)$.
V - set of vertices.
$\{A, B, C, D\}$
$E \subseteq V \times V$ - set of edges.
$\{\{A,B\},\{A,B\},\{A,C\},\{B,C\},\{B,D\},\{B,D\},\{C,D\}\}$.

For CS 70, usually simple graphs.
No parallel edges.
Multigraph above.

Directed Graphs



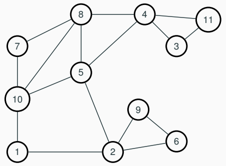
$G = (V, E)$.
V - set of vertices.
$\{1, 2, 3, 4\}$
E ordered pairs of vertices.
$\{(1,2),(1,3),(1,4),(2,4),(3,4)\}$

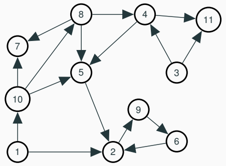
One way streets.
Tournament: 1 beats 2, ...
Precedence: 1 is before 2, ..

Social Network: Directed? Undirected?
Friends. Undirected.
Likes. Directed.

Graph Concepts and Definitions.

Graph: $G = (V, E)$
neighbors, adjacent, degree, incident, in-degree, out-degree





Neighbors of 10? 1,5,7, 8.
u is neighbor of v if $(u,v) \in E$.
Neighbors: All vertices that are adjacent to a vertex.
Edge $(10,5)$ is incident to vertex 10 and vertex 5.
Edge (u,v) is incident to u and v.
Degree of vertex 1? 2
Degree of vertex u is number of incident edges.
Equals number of neighbors in simple graph.

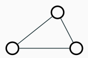
Directed graph?
In-degree of 10? 1 Out-degree of 10? 3

Quick Proof.

The sum of the vertex degrees is equal to

(A) the total number of vertices, $|V|$.
(B) the total number of edges, $|E|$.
(C) What?

Not (A)! Triangle.



Not (B)! Triangle.

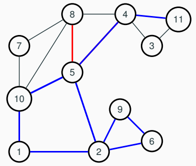
What? For triangle number of edges is 3, the sum of degrees is 6.

Could it always be...$2|E|$?

How many incidences does each edge contribute? 2.
$2|E|$ incidences are contributed in total!
What is degree v? incidences contributed to v!
sum of degrees is total incidences ... or $2|E|$.

Thm: Sum of vertex degress is $2|E|$.

Paths, walks, cycles, tour.

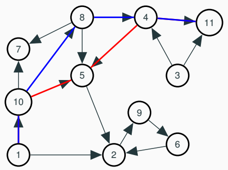


A path in a graph is a sequence of edges.

Path? $\{1,10\}$, $\{8,5\}$, $\{4,5\}$? No!
Path? $\{1,10\}$, $\{10,5\}$, $\{5,4\}$, $\{4,11\}$? Yes!
Path: $(v_1,v_2),(v_2,v_3),\ldots(v_{k-1},v_k)$.
Quick Check! Length of path? k vertices or $k-1$ edges.
Cycle: Path with $v_1 = v_k$. Length of cycle? $k-1$ vertices and edges!
Path is usually simple. No repeated vertex!
Walk is sequence of edges with possible repeated vertex or edge.
Tour is walk that starts and ends at the same node.

Quick Check!
Path is to Walk as Cycle is to ?? Tour!

Directed Paths.



Path: $(v_1, v_2), (v_2, v_3), \ldots (v_{k-1}, v_k)$.
Paths, walks, cycles, tours ... are analagous to undirected now.

Thank you!

Congrats on surviving the first week!

Have a good weekend!

Don't forget your homework, homework party tonight.