A Random Walk through CS70, Pt. III: Number Theory, Polynomials, etc.

CS70 Summer 2016 - Lecture 8D

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11 August 2016

UC Berkeley

Today

Last lecture!

Fun with number theory and polynomials.

Again, slides marked with a * are totally optional “fun stuff”.

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Covered in more detail in M115.

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Division: multiplication by multiplicative inverse. How do we find MI? EGCD!

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1. If $y = 0$, return $(x,1,0)$: $x = 1x + 0y$.
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How do we find multiplicative inverse? Solve $ax + bm = 1$.

Exponentiation in Modular Arithmetic

Repeated squaring!

$51^{43} \equiv 51^{32} \cdot 51^{8} \cdot 51^{2} \cdot 51^{1} \equiv (60) * (53) * (60) * (51) \equiv 2 \pmod{77}.$

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Immediate corollary: Fermat's little theorem. Suppose p is prime. Then $a^p \equiv a \pmod{p}$. Furthermore, if $p \not| a$, then $a^{p-1} \equiv 1 \pmod{p}$.

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Answer 2: Divide colorings into equivalence classes; two colorings are equivalent if I can get from one to the other by performing a shift. All colorings in class must be different. Why? If I can shift by some number smaller than p to get back to my original result, that means that either the coloring isn't monochromatic, or that p isn't a prime! Size of each class is p since we can shift p ways. That means $a^p - a$ must be a multiple of p!

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Let $A_1,\ldots,A_n,B_1,\ldots,B_n$ be numbers in $\{0,\ldots,p-1\}$ for some prime number p. At least one of them is not zero. We pick $w_1,\ldots,w_n$, where each w_i is picked from the set $\{0,\ldots,p-1\}$ uniformly at random. Let $\alpha = \sum_i w_i A_i$ and $\beta = \sum_i w_i B_i$. You may assume at least one of the A_is and at least one of the B_is are nonzero.

1. **(11 points)** What is the probability that $\alpha = 0 \pmod{p}$?
2. **(11 points)** Give a strictly positive (non zero) lower bound to the probability that $\alpha \cdot \beta$ is not equal to zero. (Hint: union bound)

Dot Product over Finite Fields, Solution

Part 1:

- Case 1: *Two or more A_i's are non-zero.* Look at the coefficient i of one of the non-zero ones. In order to make the sum non-zero, $w_i A_i$ must be equal to $S = \sum_{j \neq i} w_j A_j$. Therefore, we are asking for the probability that $w_i A_i = S$, which is $1/p$.

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$$\Pr[\alpha\beta = 0] = Pr[\alpha = 0 \cup \beta = 0] \leq Pr[\alpha = 0] + Pr[\beta = 0] = \frac{2}{p}$$

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Pick random a. Check if $a^{p-1} \equiv 1 \pmod{p}$. No? then composite. Yes? Prime or Carmichael w.p. at least 1/2.

Public Key Encryption, In General...

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Like this stuff? Want to learn more? CS276.

Chinese Remainder Theorem

For two congruences: Suppose $\gcd(m, n) = 1$. Then the two equations $x \equiv a \pmod{m}$ and $x \equiv b \pmod{n}$ have a unique solution mod mn

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Expand to more congruences to get CRT! Let $m_1, ..., m_k$ be relatively prime numbers. Then the k equations $x \equiv a_1 \pmod{m_1}$, ..., $x \equiv a_k \pmod{m_k}$ have a unique solution mod $m_1 m_2 ... m_k$.

Euler's Criterion and Square Roots

Theorem (Euler's Criterion): Suppose p is an odd prime and a is some integer relatively prime to p. Then $a^{(p-1)/2}$ is 1 (mod p) if and only if there exists some integer x such that $a \equiv x^2 \pmod{p}$ and -1 otherwise.

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How to find the square root? If $p \equiv 3 \pmod{4}$, and the square roots exist, then square roots of a mod p are given by $\pm a^{(p+1)/4}$.

Blum Coin Flipping

How to flip a coin over the phone?

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3. Alex, armed with knowledge of p, q, computes the square roots $\pm x, \pm y$ of a, mod n, and sends one to Grace.

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3. Alex, armed with knowledge of p, q, computes the square roots $\pm x, \pm y$ of a, mod n, and sends one to Grace.
4. If Grace got $\pm x$, then she says Alex guessed correctly. Otherwise, if she gets $\pm y$, she can factor n (since $pq|(x+y)(x-y)$) and use that to prove that she won.

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This material is covered in much greater depth in M113.

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Points to coefficients? Lagrange interpolation:

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Sum these for all i.

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Or set up the Vandermonde matrix and solve.

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^d \\ 1 & x_2 & x_2^2 & \dots & x_2^d \\ 1 & x_3 & x_3^2 & \dots & x_3^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{d+1} & x_{d+1}^2 & \dots & x_{d+1}^d \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{d+1} \end{bmatrix}$$

Secret Sharing

1. Pick some prime $q > s, n$. We will operate in $GF(q)$.
2. Pick a degree-$k-1$ polynomial P such that $P(0) = s$, i.e. $P(x) = s + a_1 x + a_2 x^2 + \ldots + a_{k-1} x^{k-1}$, where $a_1, \ldots, a_{k-1}$ are chosen randomly.
3. Give $P(i)$ to the ith official.
4. To recover the secret: have k people get together and interpolate to find $P(0)$.

No information can be recovered with less than k people if done over a prime field!

Take original message $(1, m_1), (2, m_2), ..., (n, m_n)$ in $GF(q)$ and then interpolate a polynomial.

Erasure Codes

Take original message $(1, m_1), (2, m_2), ..., (n, m_n)$ in $GF(q)$ and then interpolate a polynomial.

Send k extra points. If k drop, it's ok! Just interpolate and evaluate.

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1. Alex interpolates a degree $n-1$ polynomial $P(x)$ over the messages, like for erasure codes.

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More on codes: EE121, EE229AB.

Application/Research: PIT and Schwartz-Zippel*

Theorem (Schwartz-Zippel Lemma) : Let $Q(x_1, \ldots, x_n)$ be a *multivariate* polynomial of *total degree* d (i.e. the sum of the powers of all the variables in a term are at most d) over some field F.

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□

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Theorem (Edmonds): Let A be the matrix obtained from a bipartite graph $G = (U, V, E)$ as follows:

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Proof sketch: based on definition of determinant:

$$\det A = \sum_{\text{permutations } \pi} \text{sign}(\pi) A_{1,\pi(1)} A_{2,\pi(2)}, \ldots, A_{n,\pi(n)}$$

Zero in each term if there is no perfect matching (missing edge), nonzero otherwise. No cancellations because no two terms have same set of variables. □

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Conclusion

We hope you've enjoyed this semester and learned a lot.

Before CS70:

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Thanks for taking CS70!