A Random Walk through CS70, Pt. II: Probability

CS70 Summer 2016 - Lecture 8C

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UC Berkeley

Today

Same as yesterday (and tomorrow). Review, applications, gigs, cool examples, research questions...

Probability today!

Fundamentals

Map of outcomes in a probability space Ω to values in $[0,1]$:
$\sum_{\omega \in \Omega} \Pr[\omega] = 1$

Events: set of outcomes. $\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$.

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Inclusion-Exclusion: $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$.

Union bound: $\Pr[A_1 \cup A_2 \cup \ldots \cup A_n] \leq \Pr[A_1] + \Pr[A_2] + \ldots \Pr[A_n]$.

Total probability: if $A_1, \ldots, A_n$ partition the entire sample space (disjoint, covers all of it), then $\Pr[B] = \Pr[B \cap A_1] + \ldots + \Pr[B \cap A_n]$.

Conditional Probability

Definition:

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} \quad .$$

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From definition: $\Pr[A \cap B] = \Pr[A]\Pr[B|A]$.

Or, generally: $\Pr[A_1 \cap \ldots \cap A_n] = \Pr[A_1]\Pr[A_2|A_1]\ldots\Pr[A_n|A_1 \cap \ldots \cap A_{n-1}]$.

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Useful theorem for inference (updating beliefs). Heavily used in AI. CS188.

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Example: Random-SAT

Let's say I have some Boolean clause that looks like this ("3-CNF")

$$(a \vee b \vee \bar{c}) \wedge (\bar{b} \vee d \vee e) \wedge \ldots$$

n clauses (three boolean variables, some may be negated). What is expected number of clauses that I satisfy with a random assignment?

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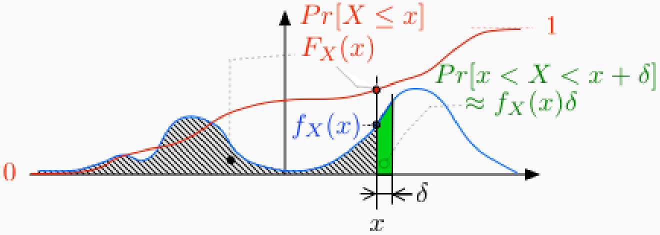
"Hardness of approximation". Ongoing topic of research.



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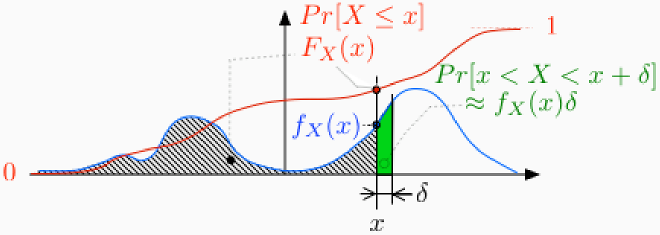

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$$\Pr[X \in [a,b]] = \int_a^b f_X(t)dt = F_X(b) - F_X(a)$$

Expectation/Variance for Continuous

Sum $\rightarrow$ Integral. Most properties carry over.

$$E[X] = \int_{-\infty}^{\infty} t f_X(t) dt$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(t) f_X(t) dt$$

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Application: Streaming Algorithm for Counting Uniques

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So PDF is $f(x) = n(1-x)^{n-1}$. Expectation: $\int_0^1 xn(1-x)^{n-1}dx = 1/(n+1)$.

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For instance: What's the distribution of the sum of two independent binomial random variables? What's the distribution of the minimum of two independent geometric random variables? Prove these formally for practice!

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Chernoff: Family of exponential bounds for sum of mutually independent 0-1 random variables. Derive by noting that $\Pr[X \geq a] = \Pr[e^{tX} \geq e^{ta}]$, and then applying Markov to bound

$$\Pr[e^{tX} \geq e^{ta}] \leq \frac{E[e^{tX}]}{e^{ta}}$$

for a good value of t.

Law of Large Numbers and CLT

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With many i.i.d. samples we converge not only to the mean, but also to a normal distribution with the same variance.

CLT: Suppose $X_1, X_2, \ldots$ are i.i.d. random variables with expectation μ and variance σ^2. Let

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This is an approximation, not a bound.

Markov Chains

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Hitting time: How long does it take us to get to some state j? Strategy: let $\beta(i)$ be the time it takes to get to j from i, for each state i. $\beta(j) = 0$. Set up system of linear equations and solve.

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Ergodic state: aperiodic + recurrent.

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Theorem: If a transient state j is accessible from state i, then state i is transient.

Proof: Suppose i is not accessible from j. Then there is a nonzero probability that, starting at i, we will go to j, at which point we will never be able to see i again. So i is transient.

On the other hand, suppose i is accessible from j. Suppose for contradiction that i is recurrent. Then if we're at j, we *have* to hit i again (because i is recurrent, so we have to go back to i if we go from i to j). But when we're at i, we know that we're definitely going to hit j sometime (because there's a nonzero chance of going to j from i, and we'll be back at i infinitely many times due to it being recurrent).So j is recurrent. Contradiction! So i has to be transient.

Markov Chain Classifications

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Ergodic Markov chain: every state is ergodic. Any finite, irreducible, aperiodic Markov chain is ergodic.

Stationary Distributions

Distribution is unchanged by state. Intuitively: if I have a lot (approaching infinity) of people on the same MC: the number of people at each state is constant (even if the individual people may move around).

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To find limiting distribution? Solve **balance equations**: $\pi = \pi P$.

Let $r_{i,j}^t$ be the probability that we first (if $i = j$, we don't count the zeroth timestep) hit j exactly t timesteps after we start at i. Then $h_{i,j} = \sum_{t \geq 1} t r_{i,j}^t$.

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Suppose we are given a finite, irreducible, aperiodic Markov chain. Then:

- There is a unqiue stationary distribution π.
- For all j, i, the limit $\lim_{t \to \infty} P_{j,i}^t$ exists and is independent of j.
- $\pi_i = \lim_{t \to \infty} P_{j,i}^t = 1/h_{i,i}$

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Cover time (expected time that it takes to hit all the vertices, starting from the worst vertex possible): bounded above by $4|V||E|$.

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Suppose that the expected return of each machine is negative - I get less money than I put in... the house always wins, after all. If I play machine 1 for a while, I expect to end up broke. Same with machine 2.

So if I play machine 1 and machine 2 alternately, I should expect to end up broke too, right? Hmm...

Parrondo's Paradox II

Let's say that the slot machines work as follows:

Machine 1: Put in some money. You gain a dollar w.p. 0.49 and lose a dollar w.p. 0.51. Pretty obvious that you lose money playing this game.

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Let's say that the slot machines work as follows:

Machine 1: Put in some money. You gain a dollar w.p. 0.49 and lose a dollar w.p. 0.51. Pretty obvious that you lose money playing this game.

Machine 2: You put in d dollars.

- Case A: If d is a multiple of 3 then you gain a dollar w.p. 0.09 and lose a dollar w.p. 0.91.
- Case B: Otherwise, you gain a dollar w.p. 0.74 and lose a dollar w.p. 0.26.

What's the probability of winning a round?

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What's the probability of winning a round? 1/3 probability of case A happening, so it would be

$$\frac{1}{3}(0.09) + \frac{2}{3}(0.74) = \frac{157}{300} > \frac{1}{2}$$

right?

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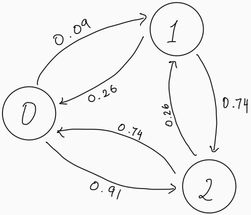
Parrondo's Paradox III

So how often do we end up with case A? Here's the approach: one state for each value of $d \pmod 3$.



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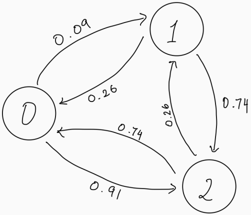

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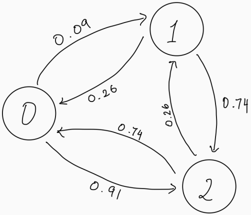

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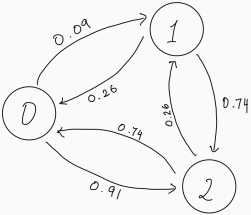

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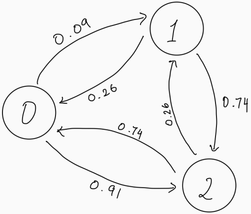

Aperiodic? Irreducible? Yep! Limiting distribution = stationary distribution! Just solve for the stationary distribution with $\pi = \pi P$.

Result: $\pi = [0.382604, 0.154728, 0.462668]$. Plug in:

$$0.3826(0.09) + (0.1547 + 0.4627)(0.74) = 0.4913 < \frac{1}{2}$$

So I lose money in the long run.

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I have d dollars... if d is a multiple of 3, probability of winning is:

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If d isn't a multiple of 3, probability of winning is:

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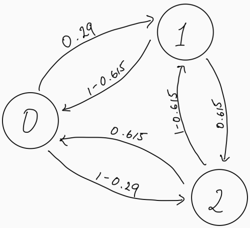

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Did we just break linearity of expectation? No! It doesn't make a whole lot of sense to talk about "expected winnings" for a state without taking into account the current state. Our distribution across states changes between the two games!

Questions?