A Random Walk through CS70, Pt. II: Probability

CS70 Summer 2016 - Lecture 8C

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Today

Same as yesterday (and tomorrow). Review, applications, gigs, cool examples, research questions...

Probability today!

Fundamentals

Map of outcomes in a probability space Ω to values in $[0,1]$: $\sum_{\omega \in \Omega} \Pr[\omega] = 1$

Events: set of outcomes. $\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$.

Inclusion-Exclusion: $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$.

Union bound: $\Pr[A_1 \cup A_2 \cup \ldots \cup A_n] \leq \Pr[A_1] + \Pr[A_2] + \ldots \Pr[A_n]$.

Total probability: if $A_1, \ldots, A_n$ partition the entire sample space (disjoint, covers all of it), then $\Pr[B] = \Pr[B \cap A_1] + \ldots + \Pr[B \cap A_n]$.

Conditional Probability

Definition:

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} .$$

Live demo.

From definition: $\Pr[A \cap B] = \Pr[A]\Pr[B|A]$.

Or, generally: $\Pr[A_1 \cap \ldots \cap A_n] = \Pr[A_1]\Pr[A_2|A_1] \ldots \Pr[A_n|A_1 \cap \ldots \cap A_{n-1}]$.

Bayes' Theorem

$$\Pr[A|B] = \frac{\Pr[A]\Pr[B|A]}{\Pr[B]}$$

Or if I know for sure that exactly one of $A_1, \ldots, A_n$ hold, then:

$$\Pr[A_k|B] = \frac{\Pr[A_k]\Pr[B|A_k]}{\sum_k \Pr[A_k]\Pr[B|A_k]} .$$

Useful theorem for inference (updating beliefs). Heavily used in AI. CS188.

Random Variables: Discrete

Random variable: function that assigns a real number $X(\omega)$ to each outcome ω in a probability space.

Random variables X, Y are independent if the events $Y = a$ and $X = b$ are independent for all a, b. If X, Y independent, then $f(X)$, $g(Y)$ independent for all f, g.

Expectation: $E[X] = \sum_t t \Pr[X = t]$

Tail sum: for nonnegative r.v. X: $E[X] = \sum_{i=0}^{\infty} \Pr[X > i]$.

Expectation of function: $E[g(X)] = \sum_t g(t) \Pr[X = t]$

Variance: $Var[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$

Standard deviation: square root of variance.

Linearity of expectation: $E[aX + bY] = aE[X] + bE[Y]$

For independent RV: $E[XY] = E[X]E[Y]$, $Var[X + Y] = Var[X] + Var[Y]$

Example: Random-SAT

Let's say I have some Boolean clause that looks like this ("3-CNF")

$$(a \vee b \vee \overline{c}) \wedge (\overline{b} \vee d \vee e) \wedge \ldots$$

n clauses (three boolean variables, some may be negated). What is expected number of clauses that I satisfy with a random assignment? $7n/8$.

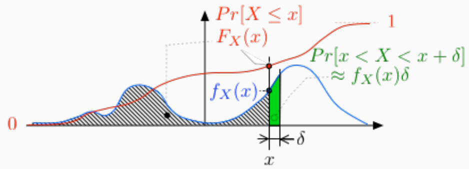
Doesn't matter if variables are repeated! Expectation is linear.

Also proves (by probabilistic method) that there exists some assignment satisfying at least 7/8 of the clauses.

Turns out that we don't know any better constant-factor approximation for this. 7/8 is the best we can do! If we can efficiently do better (i.e. $7/8+\varepsilon$ fraction of clauses satisfied, for constant ε) this would prove $P = NP$ which would, among many other things, render public key cryptography impossible!

"Hardness of approximation". Ongoing topic of research.

Random Variables: Continuous



Distributions represented with a pdf

$$f_X(t) = \lim_{\delta \to 0} \frac{\Pr[X \in [t, t+\delta]]}{\delta}$$

...or, equivalently, a cdf:

$$F_X(t) = \Pr[X \leq t] = \int_{-\infty}^{t} f_X(z)dz.$$

$$\Pr[X \in [a,b]] = \int_a^b f_X(t)dt = F_X(b) - F_X(a)$$

Expectation/Variance for Continuous

Sum $\to$ Integral. Most properties carry over.

$$E[X] = \int_{-\infty}^{\infty} t f_X(t)dt$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(t) f_X(t)dt$$

$$Var[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$$

Linearity of expectation: $E[aX + bY] = aE[X] + bE[Y]$

For independent RV: $E[XY] = E[X]E[Y]$, $Var[X+Y] = Var[X] + Var[Y]$

Application: Streaming Algorithm for Counting Uniques

Let's say that you're building a server that wants to count unique visitors. But you only have a very small amount of memory - enough to remember one number. How do you distinguish between a million unique visitors and a single IP address sending a million requests to your site?

Map each IP address to a single number between 0 and 1 uniformly.

Keep the minimum number of all the visitors to your website (only requires space for one number!).

What's the number you get? Minimum of $Uniform(0,1)$.Distribution? CDF:

$$\Pr(\min_i X_i \leq x) = 1 - \Pr(\text{all } x_i \text{ at least } x) = 1 - (1-x)^n$$

So PDF is $f(x) = n(1-x)^{n-1}$. Expectation: $\int_0^1 xn(1-x)^{n-1}dx = 1/(n+1)$. Just invert the minimum number to estimate number of unique visitors!

Distributions

Discrete: Uniform, Bernoulli, geometric, binomial, Poisson

Continuous: Exponential, normal, uniform.

Make sure you know what they mean intuitively (although formula sheet will have the formulas for them).

For instance: What's the distribution of the sum of two independent binomial random variables? What's the distribution of the minimum of two independent geometric random variables? Prove these formally for practice!

Tail Bounds

Markov: For X non-negative, a positive,

$$\Pr[X \geq A] \leq \frac{E[X]}{a}.$$

Chebyshev: For all a positive,

$$\Pr[|X - E[X]| \geq a] \leq \frac{Var[X]}{a^2}.$$

Chernoff: Family of exponential bounds for sum of mutually independent 0-1 random variables. Derive by noting that $\Pr[X \geq a] = \Pr[e^{tX} \geq e^{ta}]$, and then applying Markov to bound

$$\Pr[e^{tX} \geq e^{ta}] \leq \frac{E[e^{tX}]}{e^{ta}}$$

for a good value of t.

Law of Large Numbers and CLT

If $X_1, X_2, \dots$ are pairwise independent, and identically distributed with mean μ: $\Pr[\left|\frac{\sum_i X_i}{n} - \mu\right| \geq \varepsilon] \to 0$ as $n \to \infty$.

With many i.i.d. samples we converge not only to the mean, but also to a normal distribution with the same variance.

CLT: Suppose $X_1, X_2, \dots$ are i.i.d. random variables with expectation μ and variance σ^2. Let

$$S_n := \frac{(\sum_i X_i) - n\mu}{\sigma\sqrt{n}}$$

Then S_n tends towards $\mathcal{N}(0,1)$ as $n \to \infty$.

Or:

$$\Pr[S_n \leq a] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2} dx$$

This is an approximation, not a bound.

Markov Chains

Live Demo

Transition matrix P. Timesteps correspond to matrix multiplication: $\pi \to \pi P$.

Hitting time: How long does it take us to get to some state j? Strategy: let $\beta(i)$ be the time it takes to get to j from i, for each state i. $\beta(j) = 0$. Set up system of linear equations and solve.

State Classifications

State j is **accessible** from i: can get from i to j with nonzero probability. Equivalently: exists path from i to j.

i accessible from j and j accessible from i: i, j **communicate**.

If, given that we're at some state, we will see that state again sometime in the future with probability 1, state is **recurrent**. If there is a nonzero probability that we don't ever see state again, state is **transient**.

Every finite chain has a recurrent state.

State is **periodic** if, given that we're currently at that state, the probability that we are at that state s steps later is zero unless s divides some integer $\Delta > 1$.

Ergodic state: aperiodic + recurrent.

Example: Markov Proof

Here's a theorem and a proof of the sort that we might ask you to do on the test.

Theorem: If a transient state j is accessible from state i, then state i is transient.

Proof: Suppose i is not accessible from j. Then there is a nonzero probability that, starting at i, we will go to j, at which point we will never be able to see i again. So i is transient.

On the other hand, suppose i is accessible from j. Suppose for contradiction that i is recurrent. Then if we're at j, we *have* to hit i again (because i is recurrent, so we have to go back to i if we go from i to j). But when we're at i, we know that we're definitely going to hit j sometime (because there's a nonzero chance of going to j from i, and we'll be back at i infinitely many times due to it being recurrent).So j is recurrent. Contradiction! So i has to be transient.

Markov Chain Classifications

Irreducible Markov chain: all states communicate with every other state. Equivalently: graph representation is strongly connected.

Periodic Markov chain: any state is periodic.

Ergodic Markov chain: every state is ergodic. Any finite, irreducible, aperiodic Markov chain is ergodic.

Stationary Distributions

Distribution is unchanged by state. Intuitively: if I have a lot (approaching infinity) of people on the same MC: the number of people at each state is constant (even if the individual people may move around).

To find limiting distribution? Solve **balance equations**: $\pi = \pi P$.

Let $r_{i,j}^t$ be the probability that we first (if $i = j$, we don't count the zeroth timestep) hit j exactly t timesteps after we start at i. Then $h_{i,j} = \sum_{t \geq 1} t r_{i,j}^t$.

Suppose we are given a finite, irreducible, aperiodic Markov chain. Then:

- There is a unqiue stationary distribution π.
- For all j, i, the limit $\lim_{t\to\infty} P_{j,i}^t$ exists and is independent of j.
- $\pi_i = \lim_{t\to\infty} P_{j,i}^t = 1/h_{i,i}$

Random Walks

Markov chain on an undirected graph. At a vertex, pick edge with uniform probability and walk down it.

For undirected graphs: aperiodic if and only if graph is not bipartite.

Stationary distribution: $\pi_v = d(v)/(2|E|)$.

Cover time (expected time that it takes to hit all the vertices, starting from the worst vertex possible): bounded above by $4|V||E|$.

Example/Gig: Parrondo's Paradox

Let's say I have two slot machines. Each one takes some amount of money and then spits out some amount of money.

Suppose that the expected return of each machine is negative - I get less money than I put in... the house always wins, after all. If I play machine 1 for a while, I expect to end up broke. Same with machine 2.

So if I play machine 1 and machine 2 alternately, I should expect to end up broke too, right? Hmm...

Parrondo's Paradox II

Let's say that the slot machines work as follows:

Machine 1: Put in some money. You gain a dollar w.p. 0.49 and lose a dollar w.p. 0.51. Pretty obvious that you lose money playing this game.

Machine 2: You put in d dollars.

- Case A: If d is a multiple of 3 then you gain a dollar w.p. 0.09 and lose a dollar w.p. 0.91.
- Case B: Otherwise, you gain a dollar w.p. 0.74 and lose a dollar w.p. 0.26.

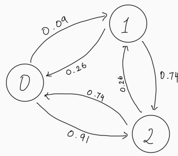
What's the probability of winning a round? 1/3 probability of case A happening, so it would be

$$\frac{1}{3}(0.09) + \frac{2}{3}(0.74) = \frac{157}{300} > \frac{1}{2}$$

right? Are you sure? No! Probability of case A happening is not 1/3! (be careful about nonuniform probability spaces. MT2 1.1/1.2!

Parrondo's Paradox III

So how often do we end up with case A? Here's the approach: one state for each value of $d \pmod 3$.



Aperiodic? Irreducible? Yep! Limiting distribution = stationary distribution! Just solve for the stationary distribution with $\pi = \pi P$.

Result: $\pi = [0.382604, 0.154728, 0.462668]$. Plug in:

$$0.3826(0.09) + (0.1547 + 0.4627)(0.74) = 0.4913 < \frac{1}{2}$$

So I lose money in the long run.

Parrondo's Paradox III

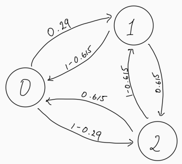
So now, what if I decide to flip a fair coin to figure out which machine to play?

I have d dollars... if d is a multiple of 3, probability of winning is:

$$\frac{1}{2}(0.49) + \frac{1}{2}(0.09) = 0.29$$

If d isn't a multiple of 3, probability of winning is:

$$\frac{1}{2}(0.49) + \frac{1}{2}(0.74) = 0.615$$



Parrondo's Paradox IV

Stationary distribution: $\pi = [0.344583, 0.254343, 0.401075]$.

Probability of winning:

$$0.3446(0.29) + (0.2543 + 0.4011)(0.615) = 0.503011 > \frac{1}{2}$$

So we expect to... gain money??!?!!!?!?!

Did we just break linearity of expectation? No! It doesn't make a whole lot of sense to talk about "expected winnings" for a state without taking into account the current state. Our distribution across states changes between the two games!

Questions?