CS70: Discrete Math and Probability

June 22, 2016

Induction

Principle of Induction.

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$$P(0) \wedge (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$$

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And we get...

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$$(\forall n \in \mathbb{N}) P(n).$$

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And we get...

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...Yes for 0,

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...Yes for 0, and we can conclude

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...Yes for 0, and we can conclude Yes for 1...

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...Yes for 0, and we can conclude Yes for 1...
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...Yes for 0, and we can conclude Yes for 1...
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And we get...

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...Yes for 0, and we can conclude Yes for 1...
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Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$

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Is predicate, $P(n)$ true for $n = k+1$?

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Statement is true for $n = 0$

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Statement is true for $n = 0$ $P(0)$ is true

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plus inductive step

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plus inductive step $\implies$ true for $n = 1$

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How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

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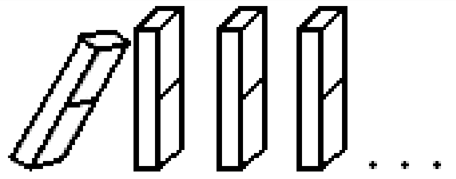
Notes visualization

Note’s visualization: an infinite sequence of dominos.



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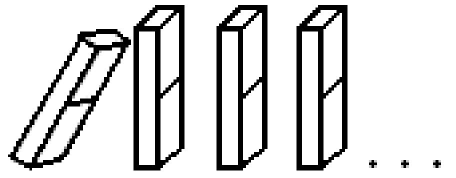

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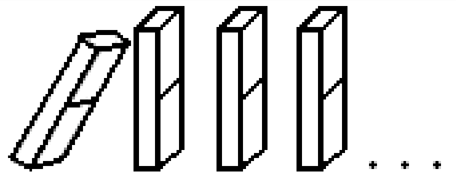

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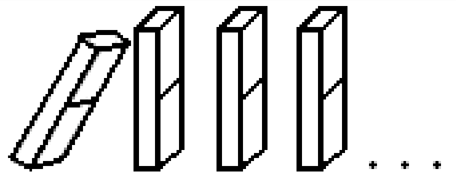

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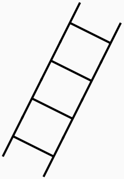

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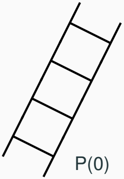

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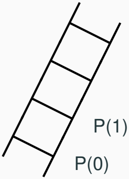

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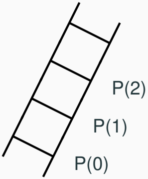

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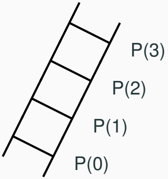

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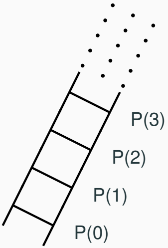

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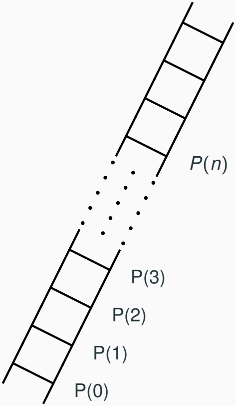

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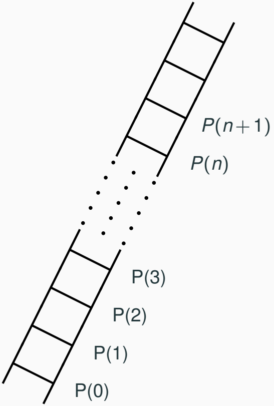

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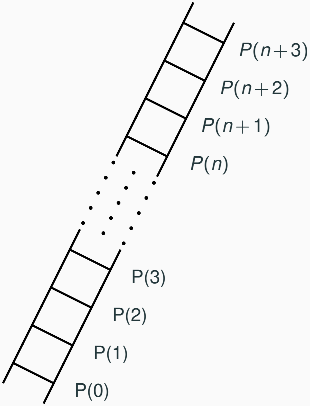

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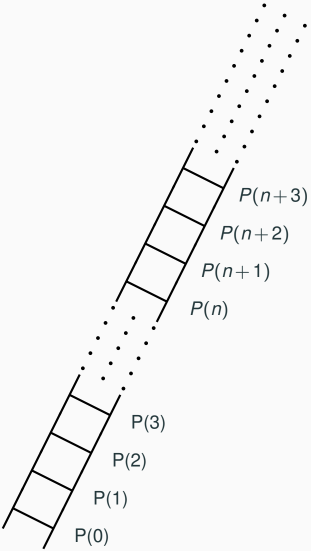

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$P(k+1)$!. By principle of induction...

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Hypothesis: $P(k)$ is true: $1+\cdots+k = \frac{k(k+1)}{2}$
Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$

$$\begin{aligned} 1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k^2+k+2(k+1)}{2} \\ &= \frac{k^2+3k+2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

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Try it yourself!

For all natural numbers n, $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$

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$$0^2+1^2+2^2\cdots+k^2+(k+1)^2 \quad = \quad (0^2+1^2+2^2\cdots+k^2)+(k+1)^2$$

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$p(k+1)$ is true.By principle of induction... □

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Another Induction Proof.

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$(k+1)^3 - (k+1)$

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$(q + k^2 + k)$ is integer (closed under addition and multiplication).

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Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\ &= 3q + 3(k^2 + k) \qquad \text{Induction Hyp.} \quad \text{Factor.} \\ &= 3(q + k^2 + k) \qquad \text{(Un)Distributive} + \text{over} \times\end{aligned}$$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).
$\implies (k+1)^3 - (k+1)$ is divisible by 3.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
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$$
\begin{aligned}
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&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
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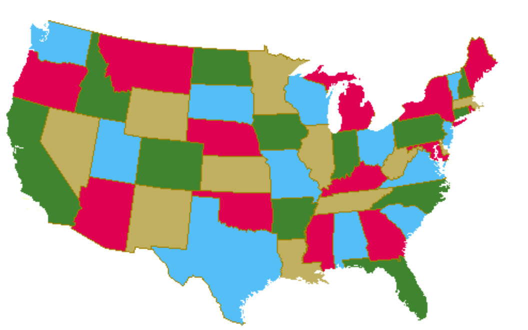
Thus, theorem holds by induction. □

Four Color Theorem.

Theorem: Any map can be 4-colored so that those regions that share an edge have different colors.



Four Color Theorem.

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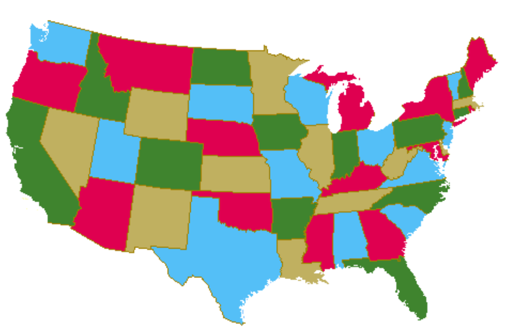

Not gonna prove it.

Two color theorem: example.

Any map formed by dividing the plane M into regions by drawing straight lines can be colored with two colors so that those regions share an edge have different colors.

Two color theorem: example.

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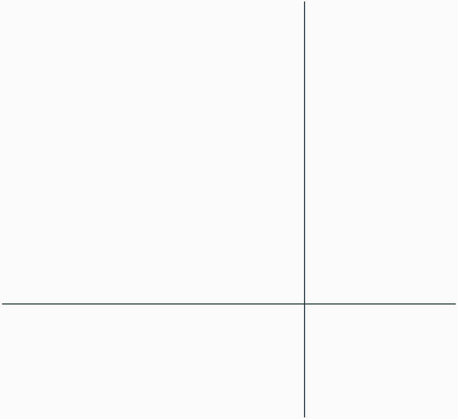

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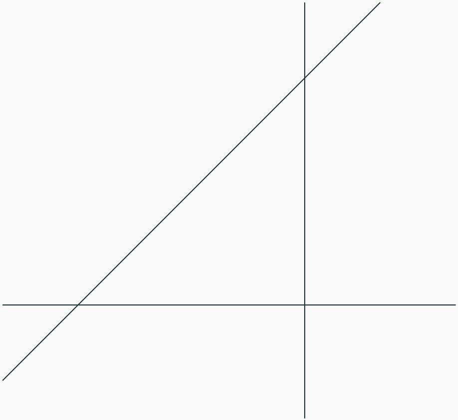

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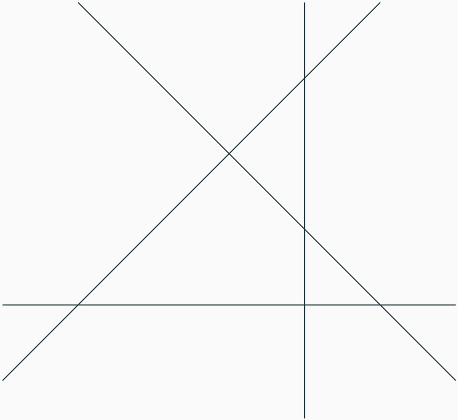

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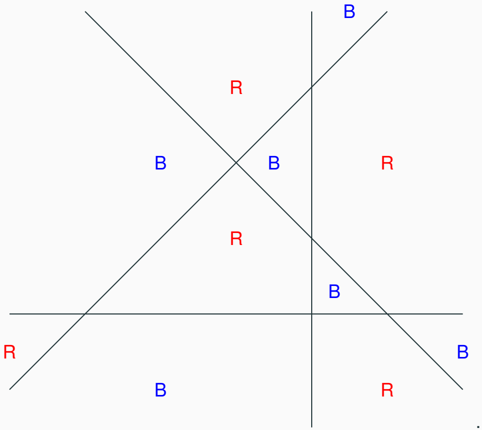

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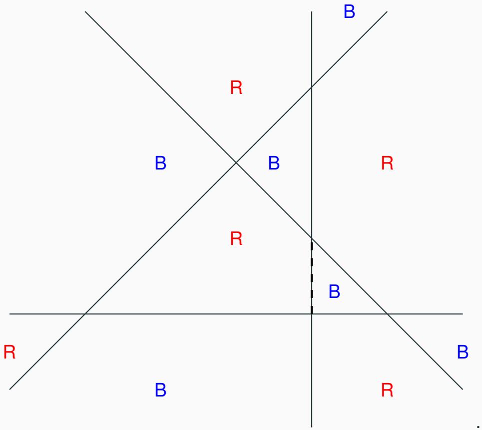

Fact: Swapping red and blue gives another valid colors.

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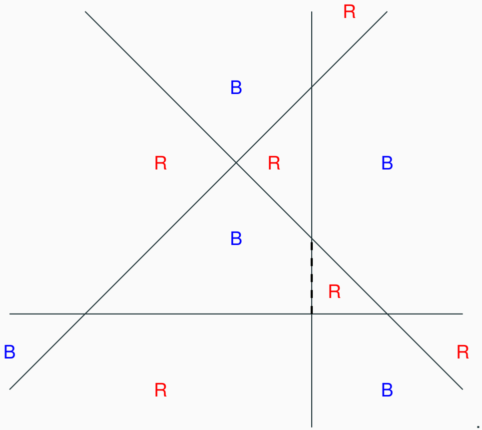
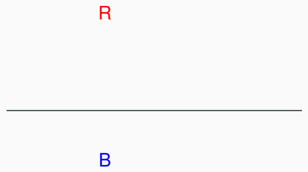


Fact: Swapping red and blue gives another valid colors.

Two color theorem: proof illustration.

Base Case.

Two color theorem: proof illustration.



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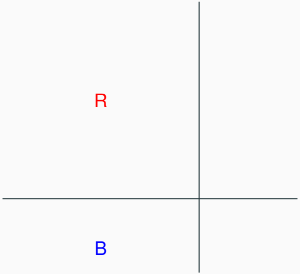
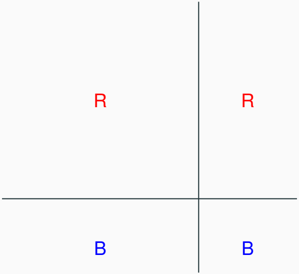


1. Add line.

Two color theorem: proof illustration.



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2. Get inherited color for split regions

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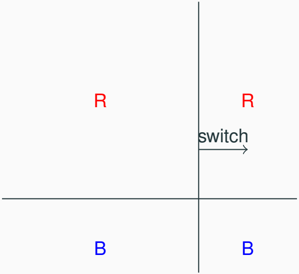

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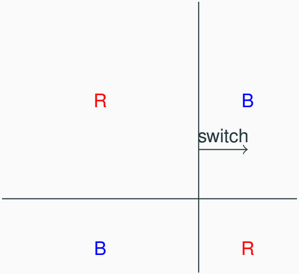
(Fixes conflicts along line, and makes no new ones.)

Two color theorem: proof illustration.



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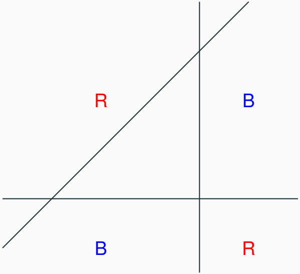

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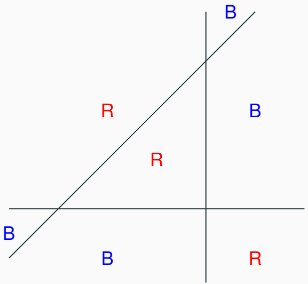

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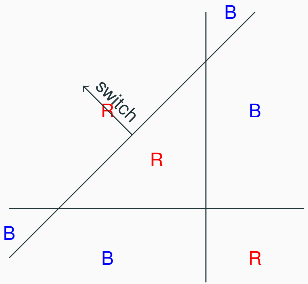

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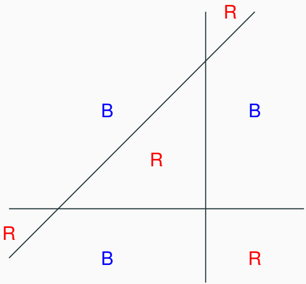

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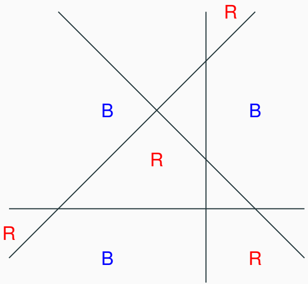

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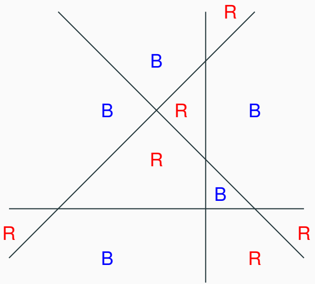

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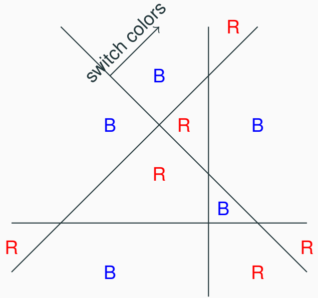

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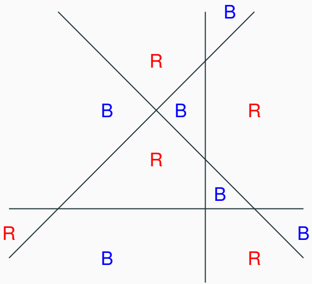

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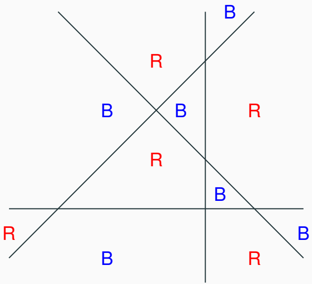

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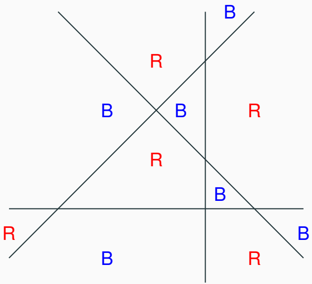

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Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

kth odd number is $2(k-1)+1$.

Base Case 1 (1th odd number) is 1^2.

Induction Hypothesis Sum of first k odds is perfect square a^2

Induction Step

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is a^2+2k+1

□

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

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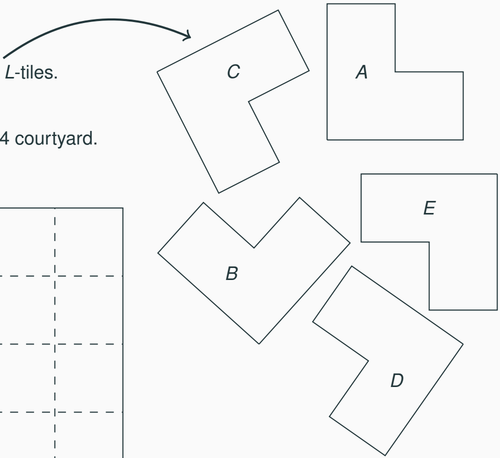
   ... P(k+1)!

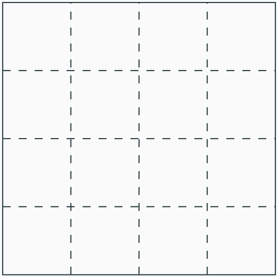
□

Tiling Cory Hall Courtyard.

Use these L-tiles.



To Tile this 4×4 courtyard.



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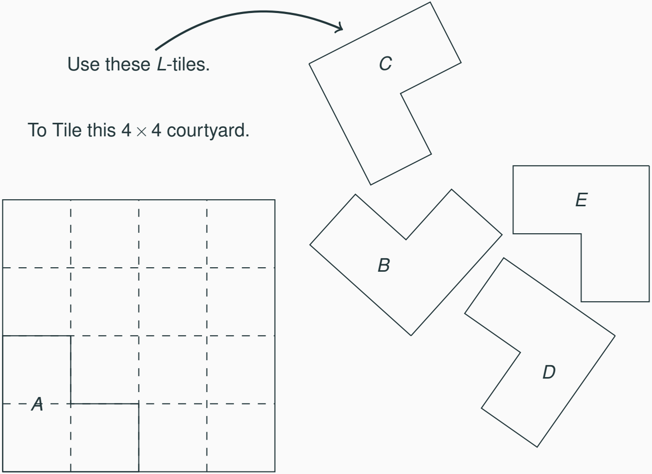

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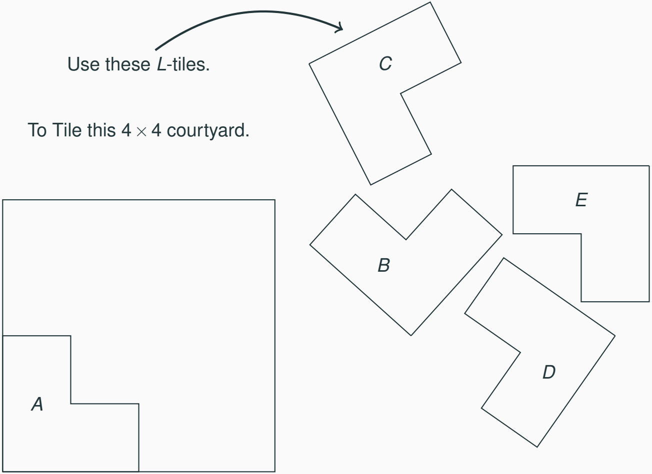

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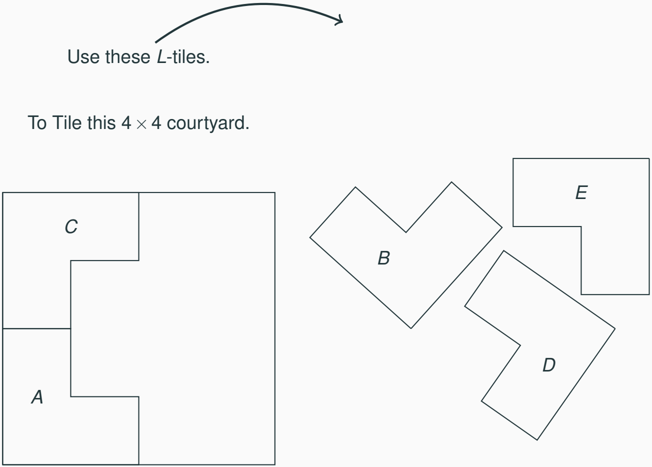

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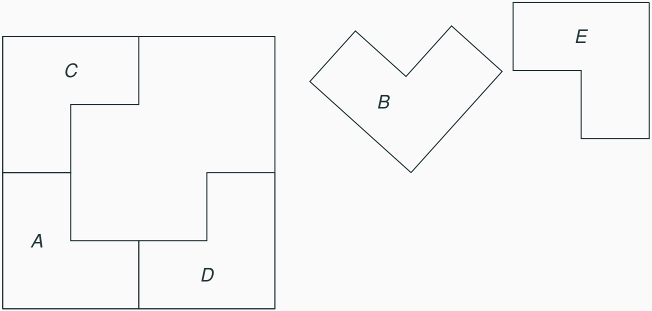

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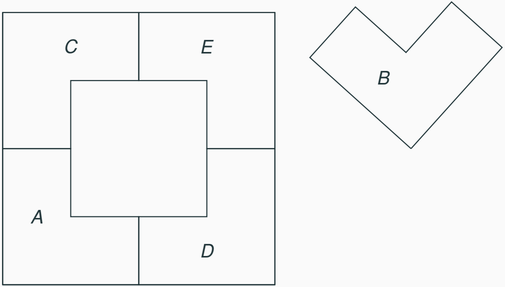

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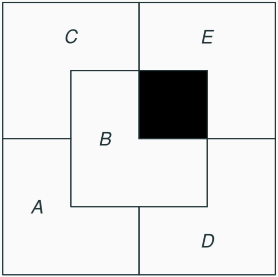

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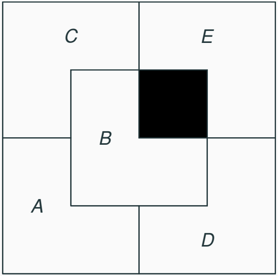

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Tiling Cory Hall Courtyard.

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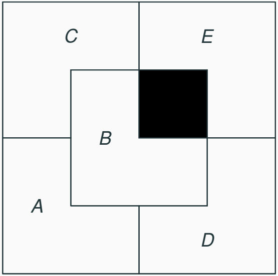


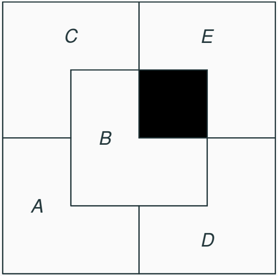
Alright!

Tiled 4×4 square with 2×2 L-tiles.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

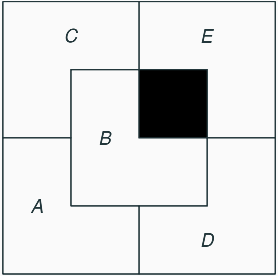
Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

Tiled 4×4 square with 2×2 L-tiles.

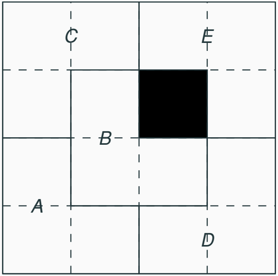
with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole)

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole) for every n!

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

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$$2^{2(k+1)} = 2^{2k} * 2^2$$

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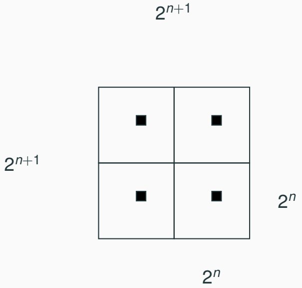

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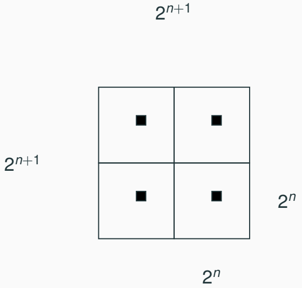

What to do now???

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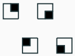

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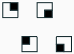

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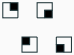

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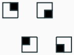

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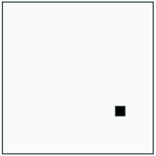

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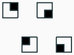

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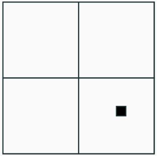

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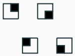

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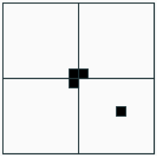

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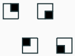

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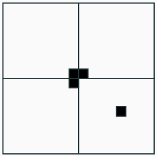

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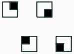

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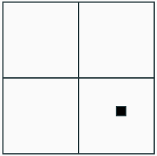

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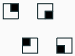

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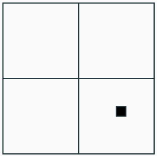

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E.g. Reduced form is “smallest” representation of a rational number a/b.

Tournaments have short cycles

Def: A **round robin tournament on n players**: every player p plays every other player q, and either $p \rightarrow q$ (p beats q) or $q \rightarrow p$ (q beats p.)

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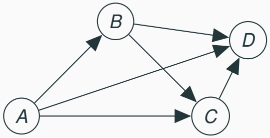

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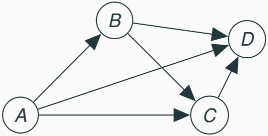

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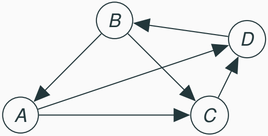

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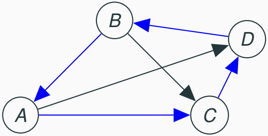

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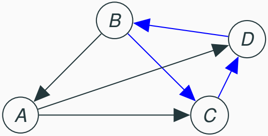

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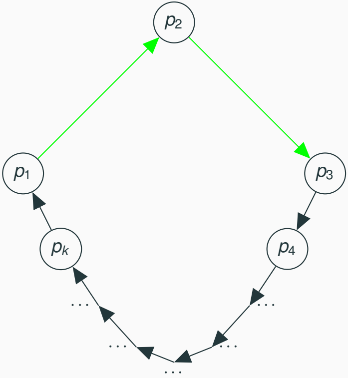

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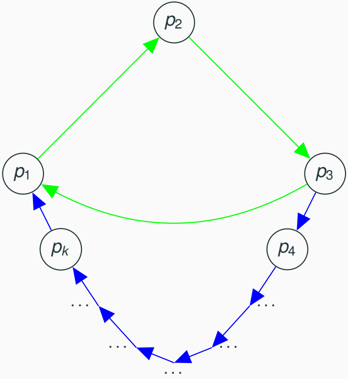

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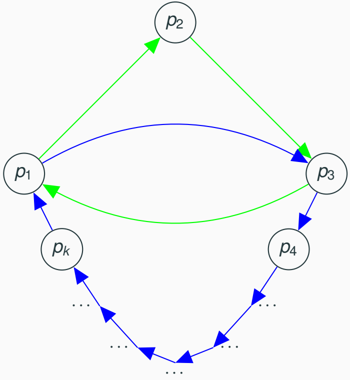

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"$p_1 \rightarrow p_3$" $\implies$ $k-1$ length cycle!

Contradiction!

Strengthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

kth odd number is $2(k-1)+1$.

Base Case 1 (1th odd number) is 1^2.

Induction Hypothesis Sum of first k odds is perfect square a^2

Induction Step

1. The $(k+1)$st odd number is $2k+1$.
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   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x',y') = find-x-y(n-4)
     return(x'+1,y')
```

Base cases: P(12) , P(13) , P(14) , P(15). Yes.

Strong Induction step:
Recursive call is correct: $P(n-4) \implies P(n)$.
$n - 4 = 4x' + 5y' \implies n = 4(x'+1) + 5(y')$

Summary: principle of induction.

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$(P(0)$

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$$(P(0) \land ((\forall k \in N)(P(k) \implies P(k+1))))$$

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$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

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$(P(0) \wedge ((\forall n \in N)(P(n)) \implies P(n+1))))$

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Also Today: strengthened induction hypothesis.

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Strengthen theorem statement.
Sum of first n odds is n^2.
Hole anywhere.
Not same as strong induction.

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Induction $\equiv$ Recursion.

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