Modular Arithmetic

CS70 Summer 2016 - Lecture 7A

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UC Berkeley

Announcements

Midterm 2 scores out.

Homework 7 is out.

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Homework 7 is out. Longer, but due next Wednesday before class, not next Monday.

There will be no homework 8.

Agenda

Some basic number theory:

- Modular arithmetic
- GCD, Euclidean algorithm, and multiplicative inverses
- Exponentiation in modular arithmetic

Mathematics is the queen of the sciences and number theory is the queen of mathematics. -Gauss

Modular Arithmetic Motivation: Clock Math

If it is 1:00 now.

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What time is it in 2 hours?

Modular Arithmetic Motivation: Clock Math

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What time is it in 2 hours? 3:00!

Modular Arithmetic Motivation: Clock Math

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What time is it in 2 hours? 3:00!

What time is it in 5 hours?

Modular Arithmetic Motivation: Clock Math

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What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

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What time is it in 15 hours?

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- What time is it in 5 hours? 6:00!
- What time is it in 15 hours? 16:00!
  - Actually 4:00.
- 16 is the “same as 4” with respect to a 12 hour clock system.

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Clock time equivalent up to to addition/subtraction of 12.

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What time is it in 100 hours?

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What time is it in 100 hours? 101:00!

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(Almost remainder, except for 12 and 0 are equivalent.)

Congruences

x is congruent to y modulo m, denoted "$x \equiv y \pmod{m}$"...

- if and only if $(x - y)$ is divisible by m (denoted $m|(x - y)$).
- if and only if x and y have the same remainder w.r.t. m.
- $x = y + km$ for some integer k.

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Congruence partitions the integers into equivalence classes ("congruence classes"). For instance, here are equivalence classes mod 7: $\{\ldots, -7, 0, 7, 14, \ldots\}$

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Theorem: If $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$, then $a + b \equiv c + d \pmod{m}$ and $a \cdot b = c \cdot d \pmod{m}$.

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Proof: Addition: $(a + b) - (c + d) = (a - c) + (b - d)$. Since $a \equiv c \pmod{m}$ the first term is divisible by m, likewise for the second term. Therefore the entire expression is divisible by m, so $a + b \equiv c + d \pmod{m}$.

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Multiplication: Let a = $k_1m + c$ and $b = k_2m + d$. Then

$$ab = (k_1m + c)(k_2m + d) = (k_1k_2m + k_1d + k_2c)m + cd$$

so $ab \equiv cd \pmod{m}$.

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When is there a solution to the equation $xy = 1 + km$?

Multiplicative Inverses: Existence

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Since $\gcd(x, m) = 1$, we must have that $m | (a-b)$, which implies that $a - b \geq m$. But $a, b \in \{0, 1, \ldots, m-1\}$, so this is impossible. Contradiction. $\square$

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I need $\min(x, m)$ divisions. For 64-bit integers, that means up to $2^64 = 18446744073709551616$ divisions - assuming one division per nanosecond (1 GHz), that's about 585 years to compute a single gcd :(

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Now suppose k divides both x and $y + ax$. Then again by lemma, it must divide $y + ax - ax = y$.

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Now suppose k divides both x and $y + ax$. Then again by lemma, it must divide $y + ax - ax = y$.

Therefore, the set of common divisors of x, y is the same as the set of divisors of $x, y + ax$ which means that the gcd must be the same as well. □

The Euclidean Algorithm

This leads to an algorithm for computing the gcd of x and y (assuming $x \geq y \geq 0$):

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How long does it take to run? $O(\log y)$ iterations. Proof: not today.

A lot faster than brute force!

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How do we find a, b?

EGCD: Motivation

Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$(3)12 + (-1)35 = 1$.

$a = 3$ and $b = -1$.

The multiplicative inverse of $12 \pmod{35}$ is 3.

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$$\gcd(35, 12) = \gcd(12, 11) = \gcd(11, 1) = \gcd(1, 0) = 1$$

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How did we get 11 from 35 and 12? $35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11$.

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What if we work backwards?

$$1 = 12 - 1(11) = 12 - 1(35 - 2(12)) = 3(12) - 1(35) \quad .$$

Just keep back-substituting.

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Inputs: $x \geq y \geq 0$ with $x > 0$. Outputs: integers (d, a, b) where $d = \text{gcd}(x, y) = ax + by$.

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3. Return $(d, b, a - b \lfloor x/y \rfloor)$.

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3. Return $(d, b, a - b \lfloor x/y \rfloor)$.

Since this is just GCD (except we track some more numbers), $d = \text{gcd}(x, y)$.

Need to show that $d = ax + by$.

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Return value: $(d, b, a - b\lfloor x/y \rfloor)$ where (d, a, b) is return value of the extended GCD algorithm on $(y, x - y\lfloor x/y \rfloor)$. By inductive hypothesis, (d, a, b) is the correct return value for the recursive call, i.e. $ay + b(x - y\lfloor x/y \rfloor) = d$.

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For the base case, $y = 0$. We return $(x, 1, 0)$ and $x = 1x + 0y$, as desired.

Now suppose for induction that extended GCD returns the correct coefficients for all y in $[0, k]$. It suffices to show the claim for $y = k + 1$.

Return value: $(d, b, a - b\lfloor x/y \rfloor)$ where (d, a, b) is return value of the extended GCD algorithm on $(y, x - y\lfloor x/y \rfloor)$. By inductive hypothesis, (d, a, b) is the correct return value for the recursive call, i.e. $ay + b(x - y\lfloor x/y \rfloor) = d$.

Therefore:

$$d = ay + b(x - y\lfloor x/y \rfloor) = ay + bx - by\lfloor x/y \rfloor = bx + (a - \lfloor x/y \rfloor b)y \quad ,$$

as desired. $\square$

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What about exponentiation? After the break.

Break!

Exponentiation: Motivation

Can we just simplify exponentiation under congruence the same way we did with addition and multiplication?

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$$2^6 \equiv 64 \equiv 4 \not\equiv 2^1 \pmod{5} .$$

Guess not.

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$$51^{32} \cdot 51^8 \cdot 51^2 \cdot 51^1 = (60) * (53) * (60) * (51) \equiv 2 \pmod{77} .$$

Repeated Squaring, Formally

To compute $x^y \pmod{n}$:

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$$x^{43} = x^{32} * x^8 * x^2 * x^1$$

.

How many multiplications required? $O(\log y)$. Much faster than multiplying y times!

Algebraic simplification?

Repeated squaring is less useful when you're dealing with symbolic expressions… what else do we have in our toolbox?

Reduced Residue Systems

Remember that we can divide up the integers into congruence classes mod n for any n.

Any set of n integers, one from each congruence class, is known a **complete residue system** mod n.

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One complete residue system mod n: $\{0, 1, 2, ..., n-1\}$.

A subset of a complete residue system only consisting of numbers relatively prime to n is called a **reduced residue system**.

One reduced residue system mod n: list of all nonnegative numbers smaller than n that are relatively prime to it (i.e. numbers whose gcd with n is 1).

Euler's Totient Function

For $n \geq 1$, the *totient function* $\phi(n)$ denotes the number of elements in any reduced residue system mod n. Equivalently: the number of nonnegative numbers smaller than n that are relatively prime to n.

Euler's Theorem (a.k.a. Euler-Fermat Theorem) I

Theorem: Suppose $\gcd(a, n) = 1$. Then $a^{\phi(n)} = 1$.

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Proof of Lemma 1: Since $\gcd(a, n) = 1$, we know that there must exist some c such that $ac \equiv 1 \pmod{n}$.

Now suppose $\{a_1, ..., a_n\}$ is a complete residue system mod n. Then for any integer d, there is a unique k such that $c(d - b) \equiv a_k \pmod{n}$.

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Therefore: $(d - b) \equiv ac(d - b) \equiv aa_k \pmod{n}$ so $d \equiv aa_k + b \pmod{n}$. So each integer is congruent with at least one element in set.

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Now suppose $d \equiv aa_j + b \pmod{n}$ and $d \equiv aa_k + b \pmod{n}$. Then $c(d - b) = aca_j = a_j = aca_k = a_k \pmod{n}$. So each integer is congruent with **exactly** one element in set. So set is a CRS. $\square$

Euler's Theorem (a.k.a. Euler-Fermat Theorem) II

Lemma 2: Suppose $\gcd(a, n) = 1$, and $\{a_1, ..., a_{\phi(n)}\}$ is a reduced residue system mod n. Then $\{aa_1, ..., aa_{\phi(n)}\}$ is also a reduced resude system mod n.

Proof of Lemma 2: Each of $\{aa_1, ..., aa_{\phi(n)}\}$ must be a distinct element in a complete residue system mod n by Lemma 1. Since a reduced residue system has $\phi(n)$ elements, it suffices to show that each of $\{aa_1, ..., aa_{\phi(n)}\}$ is relatively prime to n. But this follows immediately from the fact that both a and a_k are relatively prime to n for all k. $\square$

Euler's Theorem (a.k.a. Euler-Fermat Theorem) III

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Multiply all the elements of the sets together. They have to be the same.

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So:

$$a^{\phi(n)} \equiv 1 \pmod{n} .$$

□

Fermat's Little Theorem

Fermat's little theorem follows immediately from Euler's theorem.

Theorem: Suppose p is prime. Then $a^p \equiv a \pmod{p}$. Furthermore, if $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$.

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On the other hand, suppose $p \not| a$. How many nonnegative numbers smaller than p are relatively prime to it? $p - 1$ (all except 0). So by Euler's theorem: $a^{p-1} = a^{\phi(p)} = 1$. $\square$

Gig(ish): A Combinatorial Look at Fermat's Little Theorem

Questions?