Midterm 2 Review

CS70 Summer 2016 - Lecture 6D

Grace Dinh

28 July 2016

UC Berkeley

Midterm 2: Format

8 questions, 190 points, 110 minutes (same as MT1).

Two pages (one double-sided sheet) of **handwritten** notes.

Coverage: we will assume knowledge of all the material from the beginning of the class to yesterday, but we will only explicitly test for material seen after MT1.

We will give you a formula sheet (see MT2 logistics post on Piazza to see it). On it: all the distributions we'll expect you to know (with expectation + variance), and Chernoff bounds.

Probability Basics

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Events can be combined using standard set operations.

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Union bound: $\Pr[A_1 \cup A_2 \cup \ldots \cup A_n] \leq \Pr[A_1] + \Pr[A_2] + \ldots \Pr[A_n]$.

Total probability: if $A_1, \ldots, A_n$ partition the entire sample space (disjoint, covers all of it), then $\Pr[B] = \Pr[B \cap A_1] + \ldots + \Pr[B \cap A_n]$.

What are the probabilities?

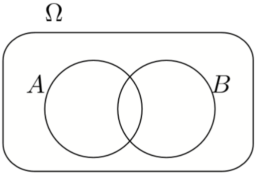

Figure : Two events

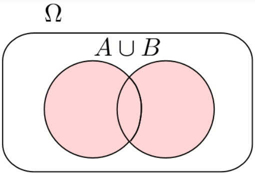

Figure : Union (or)

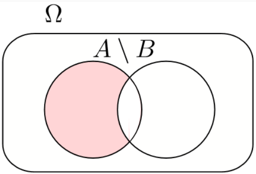

Figure : Difference (A, not B)

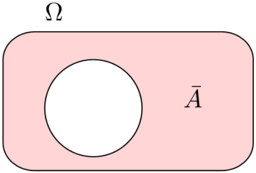

Figure : Complement (not)

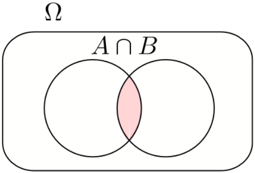

Figure : Intersection (and)

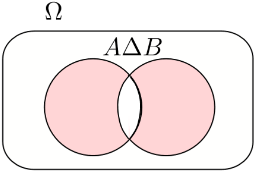

Figure : Symmetric difference (only one)

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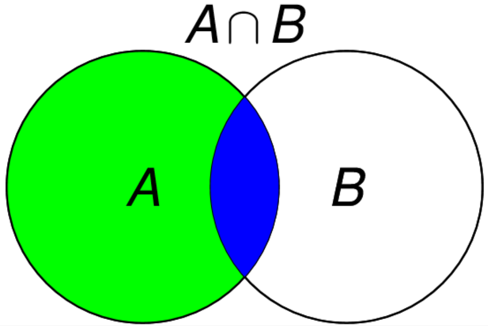

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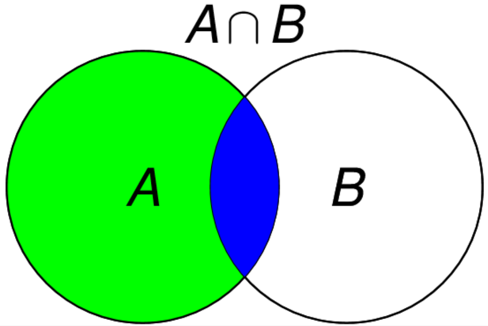

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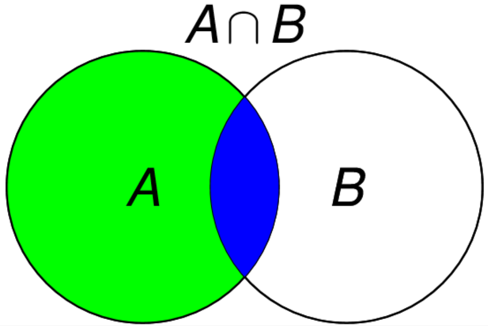

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$$\Pr[A_1 \cap \ldots \cap A_n] = \Pr[A_1] \Pr[A_2|A_1] \ldots \Pr[A_n|A_1 \cap \ldots \cap A_{n-1}] \quad .$$

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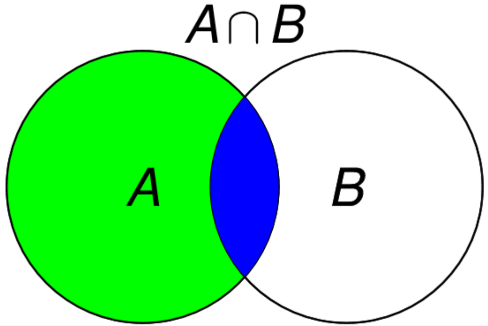

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Correlation and Independence

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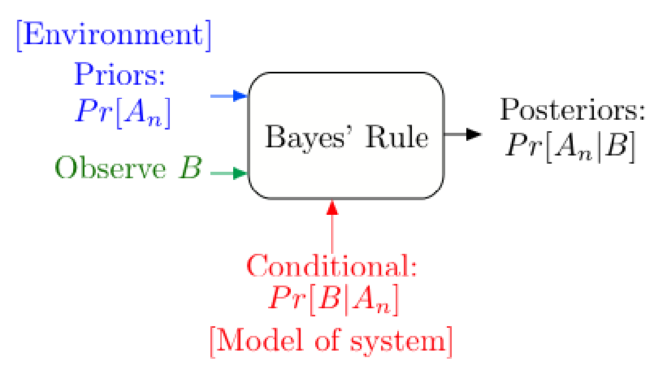
Or maybe knowing that one is true tells you that the other is likely to be true, too.

Positive Correlation: $\Pr[A \cap B] > \Pr[A]\Pr[B]$.

Negative Correlation: $\Pr[A \cap B] < \Pr[A]\Pr[B]$.

Bayes' Theorem



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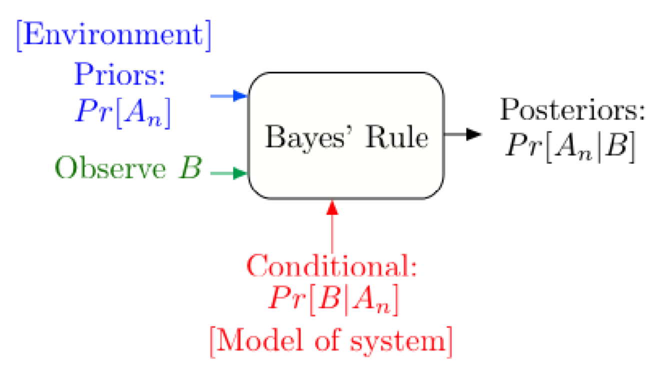

You know you will get a good grade in CS70 with some probability (prior). You take midterm 2 and get a good grade (observation). With this new information, figure out the probability that you get a good grade in CS70 (posterior).

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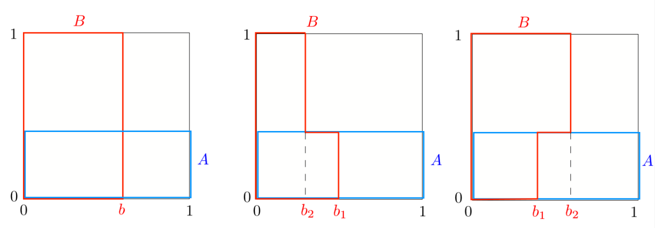

Random Variables

Fundamentals

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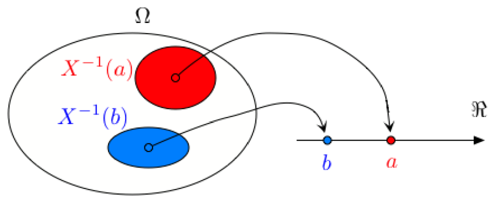
Discrete distributions: when there are a finite number of values an R.V. can take: pairs of values and probabilities. Probability of R.V. taking on a value: probability that an event that maps onto that value occurs.



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Probability space is infinite and maps onto a continuous set of reals.

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Distributions represented with a pdf

$$f_X(t) = \lim_{\delta \to 0} \frac{\Pr[X \in [t, t+\delta]]}{\delta}$$

...or, equivalently, a cdf:

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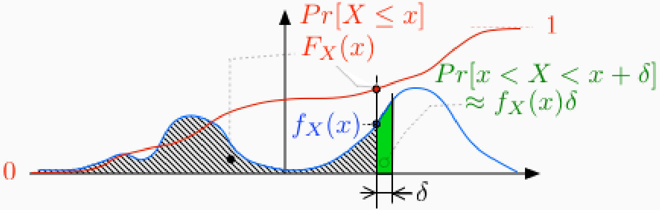

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If X, Y independent, then $f(X), g(Y)$ independent for all f, g.

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For independent X, Y: $E[XY] = E[X]E[Y]$.

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Standard deviation is defined as square root of variance.

Indicators

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Expectation? Same as probability that event happened!

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Tail Bounds and LLN

Confidence Intervals

Confidence intervals: if X falls in $[a, b]$ with probability $1 - \alpha$, then we say that $[a, b]$ is an $1 - \alpha$ confidence interval for X.

Markov

For X non-negative, a positive:

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Or: for monotone non-decreasing function f that takes non-negative values, and non-negative X:

$$\Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$$

for all a s.t. f(a) > 0.

Chebyshev

$$\Pr[|X - E[X]| \geq a] \leq \frac{Var[X]}{a^2}$$

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How did we get this? Just use Markov and use $f(x) = x^2$ as our function.

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General approach to derive these: note that

$$\Pr[X gea] = \Pr[e^{tX} \geq e^{ta}] \ .$$

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All the bounds you need are on the equation sheet on the exam.

LLN and CLT

If $X_1, X_2, \ldots$ are pairwise independent, and identically distributed with mean μ: $\Pr\left[\left|\frac{\sum_i X_i}{n} - \mu\right| \geq \epsilon\right] \to 0$ as $n \to \infty$.

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With many i.i.d. samples we converge not only to the mean, but also to a normal distribution with the same variance.

CLT: Suppose $X_1, X_2, \ldots$ are i.i.d. random variables with expectation μ and variance σ^2. Let

$$S_n := \frac{A_n - n\mu}{\sigma\sqrt{n}} = \frac{(\sum_i X_i) - n\mu}{\sigma\sqrt{n}}$$

Then S_n tends towards $\mathcal{N}(0,1)$ as $n \to \infty$.

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Or:

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This is an approximation, not a bound.

Markov Chains

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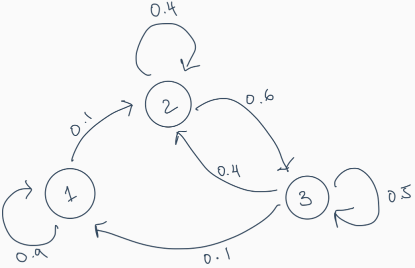
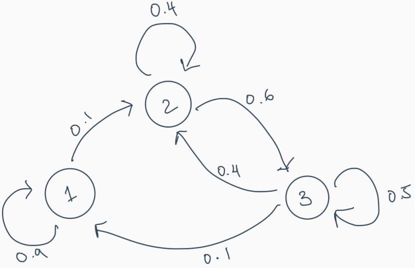


Definitions

Set of states, transition probabilities, and initial distribution.



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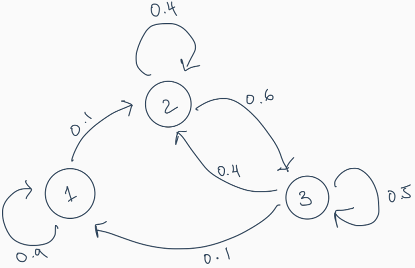

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Distributions are row vectors. Timesteps correspond to matrix multiplication: $\pi \to \pi P$.

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Set up system of linear equations and solve.

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Ergodic state: aperiodic + recurrent.

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Periodic Markov chain: any state is periodic.

Ergodic Markov chain: every state is ergodic. Any finite, irreducible, aperiodic Markov chain is ergodic.

Stationary Distributions

Distribution is unchanged by state. Intuitively: if I have a lot (approaching infinity) of people on the same MC: the number of people at each state is constant (even if the individual people may move around).

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- $\pi_i = \lim_{t \to \infty} P_{j,i}^t = 1/h_{i,i}$

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Cover time (expected time that it takes to hit all the vertices, starting from the worst vertex possible): bounded above by $4|V||E|$.

Good luck on the midterm!