Markov Chains II

CS70 Summer 2016 - Lecture 6C

Grace Dinh

27 July 2016

UC Berkeley

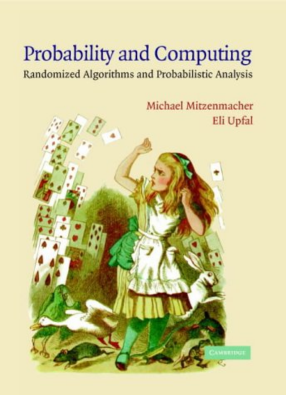
Agenda

Classification of MC states

Aperiodicity, irreducibility, ergodicity

Convergence, limiting and stationary distributions

Reference for this lecture: Ch. 7 of Mitzenmacher and Upfal, ”Probability and Computing”



Markov Chain Properties

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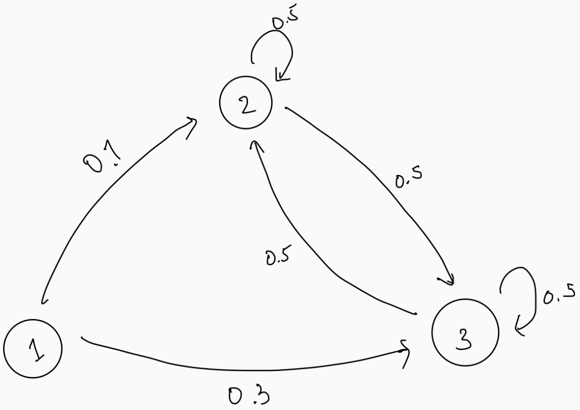
Another way of looking at it: directed connectivity. i communicates with j: exists path from i to j in the graph corresponding to the chain.

Accessibility and Communication: Example



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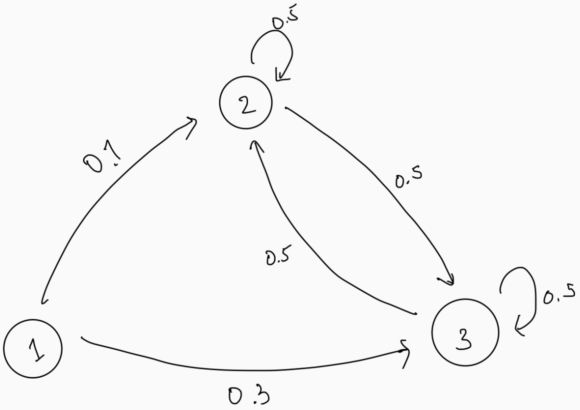

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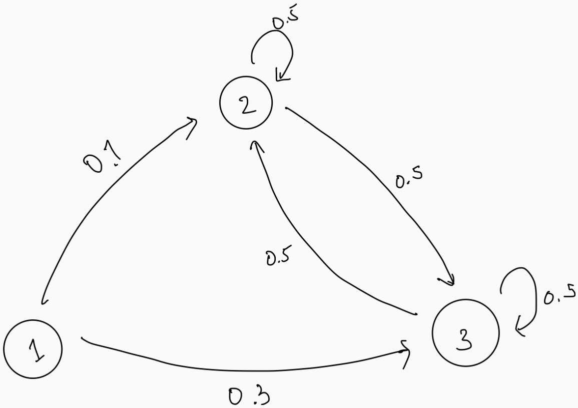

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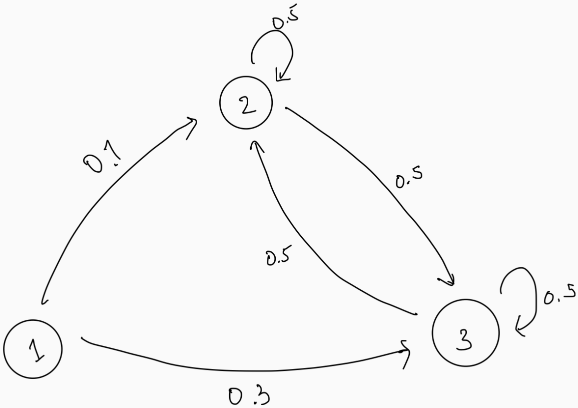

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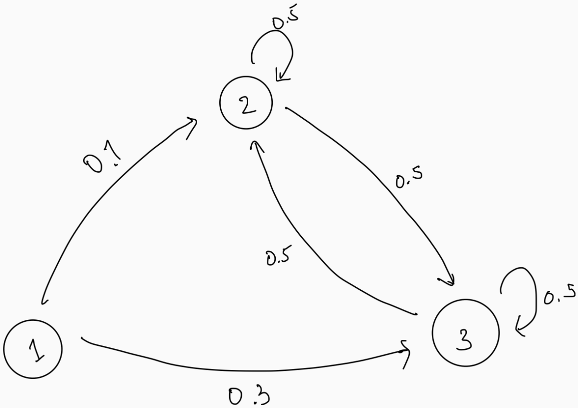

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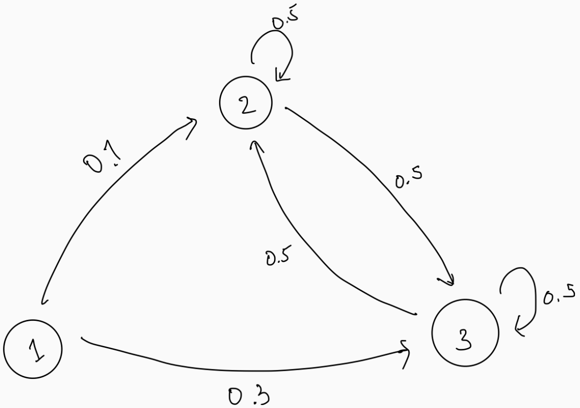

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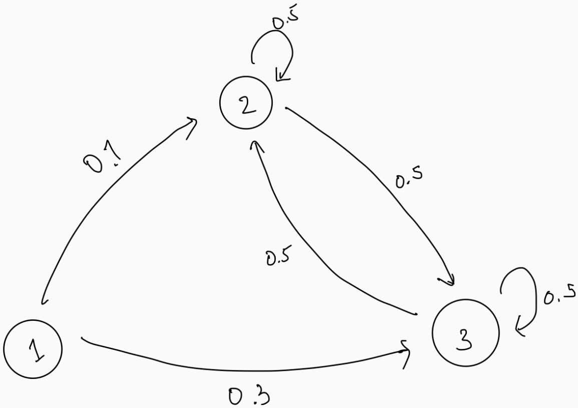

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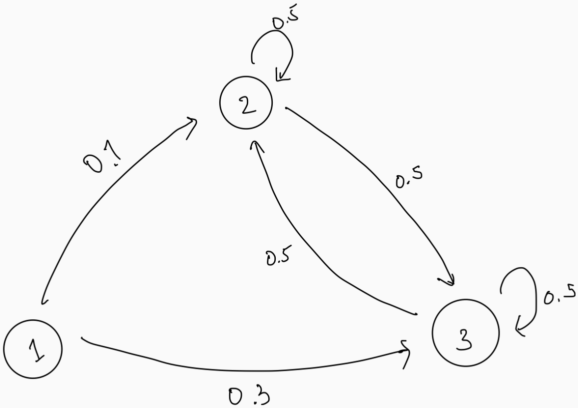

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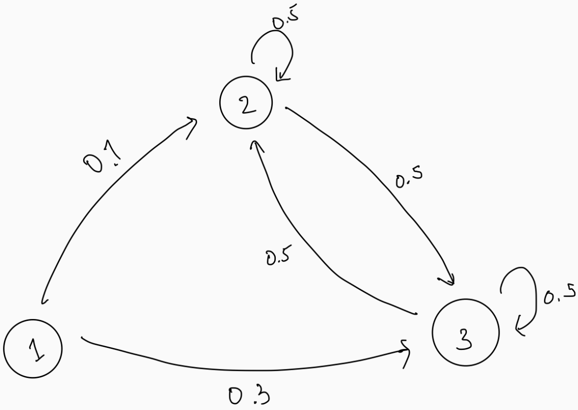

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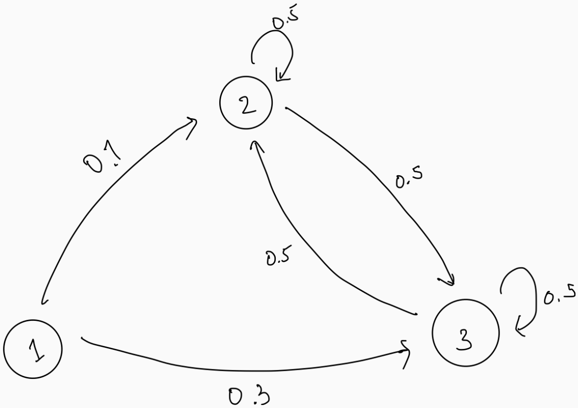

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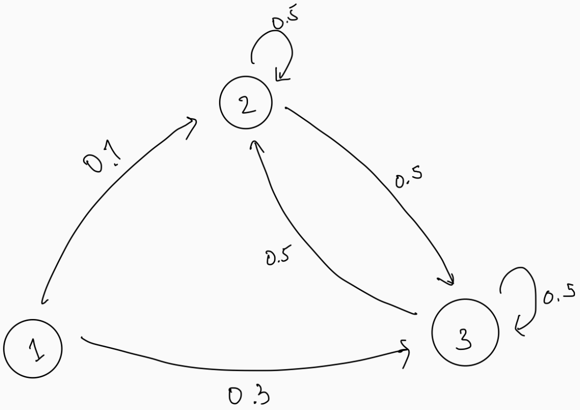

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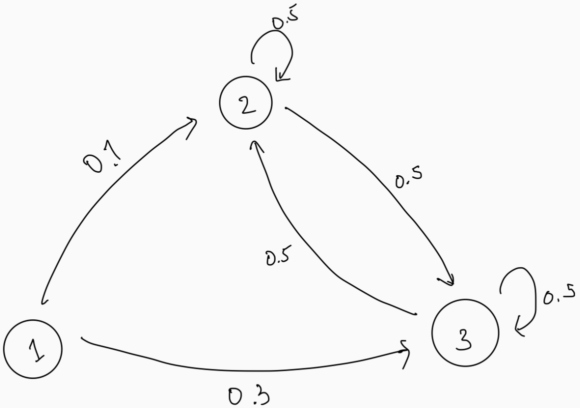

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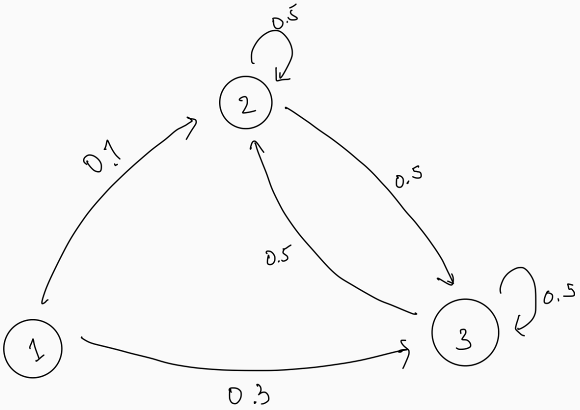

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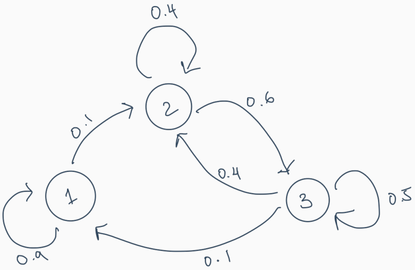

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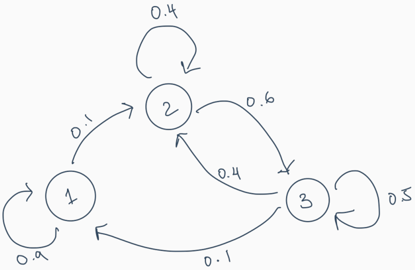

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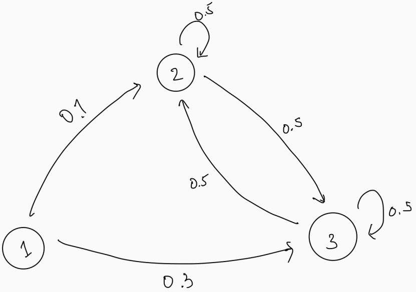

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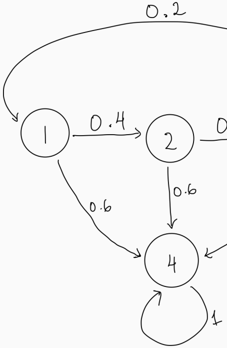

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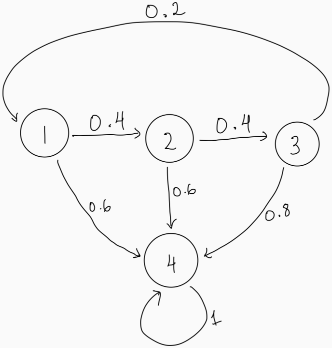

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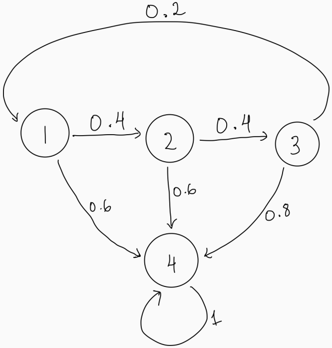

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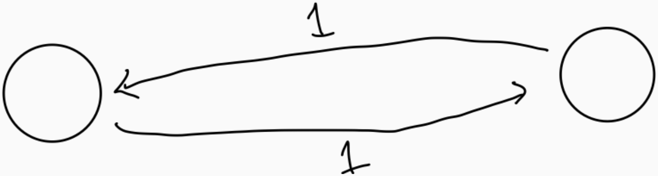

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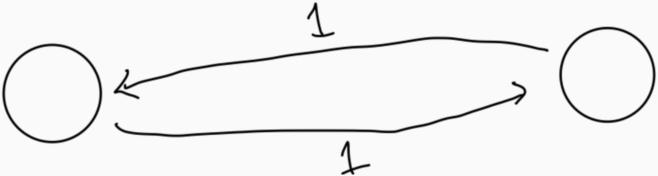

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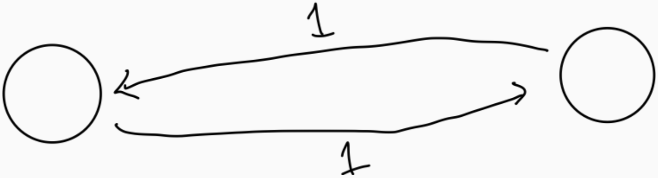

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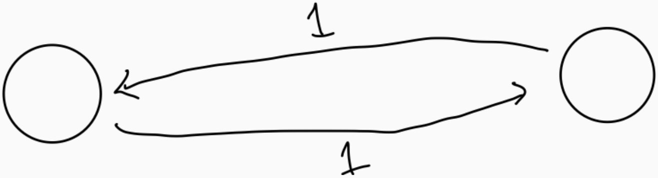

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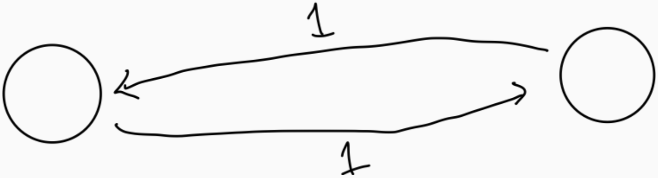

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Opposite of periodic: **aperiodic**.

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Ergodicity

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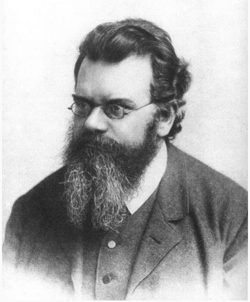

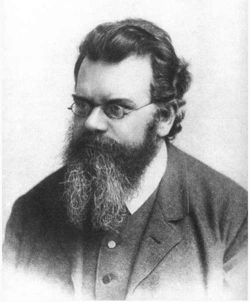

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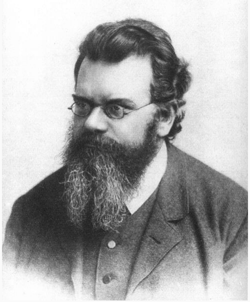

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Theorem: A finite, irreducible, aperiodic Markov chain is ergodic.

Stationary and Limiting Distributions

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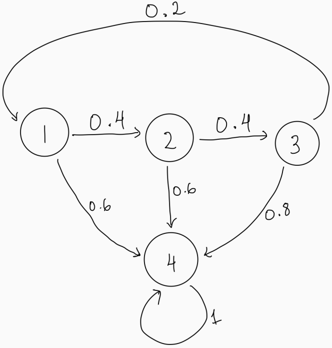

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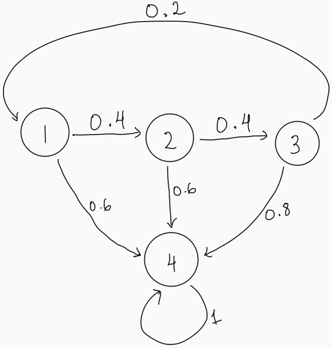

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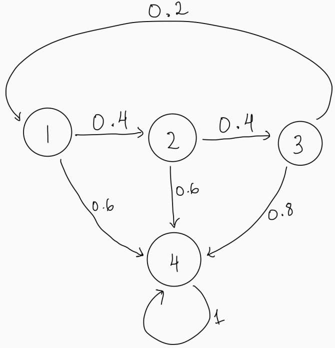

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If our distribution is $[0 \;\; 0 \;\; 0 \;\; 1]$: distribution is unchanged over a timestep.

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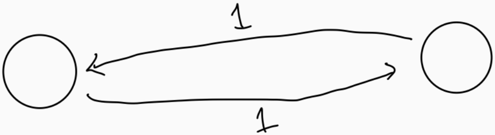

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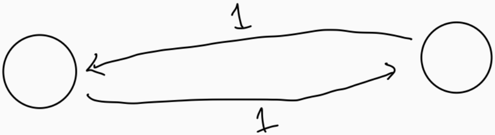

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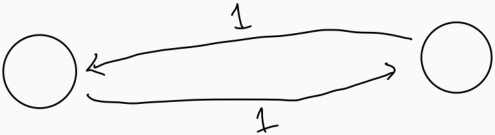

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Basically: not affected by timesteps. If we have this distribution, we have it forever.

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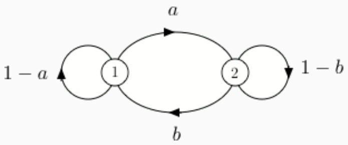
To find stationary distribution: solve $\pi P = \pi$ ("balance equations")

An Example



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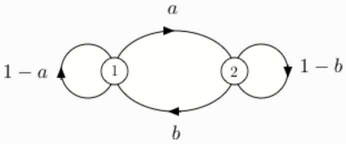

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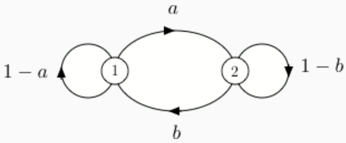

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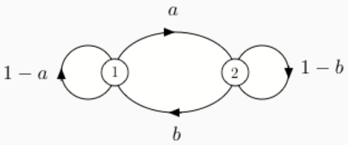

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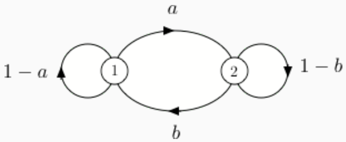

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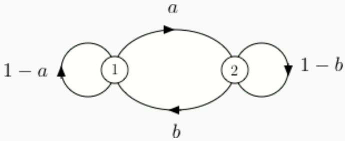

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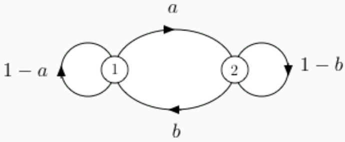

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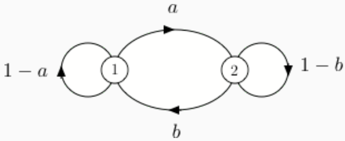

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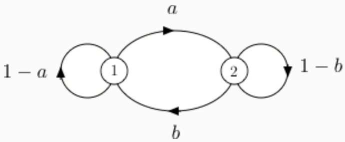

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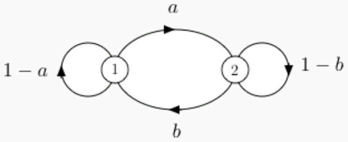

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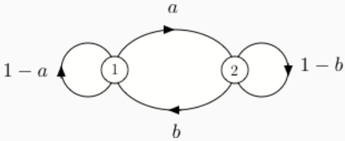

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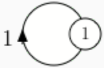

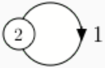

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Proof: really long and messy, see note 18 or Ch. 7 of MU. (we won't expect you to know this).

Connections between Linear Algebra and Markov Chains

It turns out that the convergence of the limiting distribution to the stationary distribution corresponds to a nice result from linear algebra: if you multiply a random vector by a matrix a lot of times, the result will converge towards an eigenvector (specifically, one corresponding to the highest eigenvalue) w.h.p.

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(No, you do not need to know this for the midterms and the homeworks).

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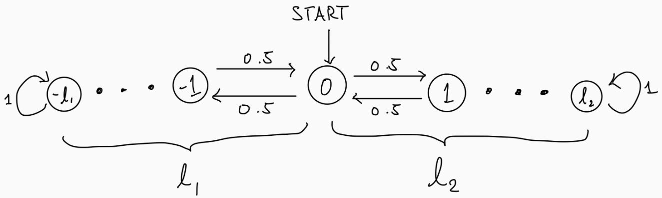

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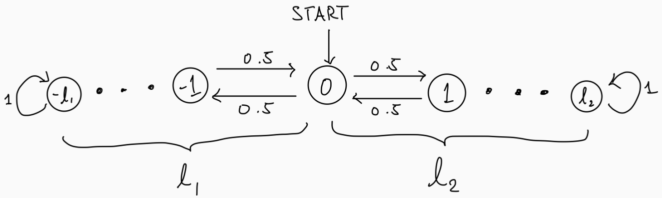

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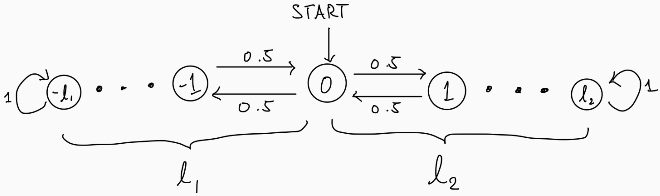

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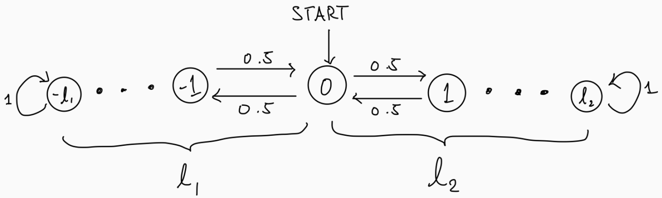

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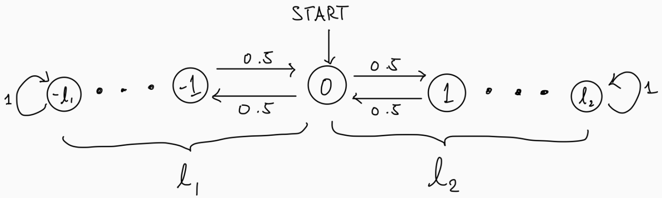

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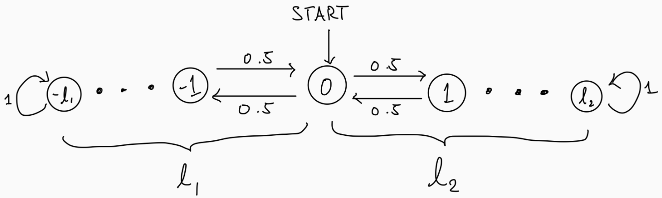

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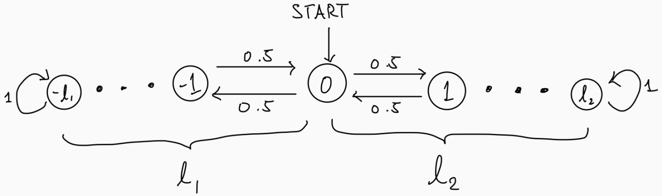

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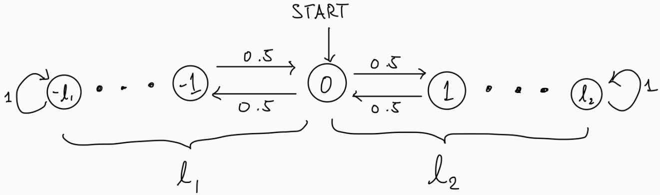

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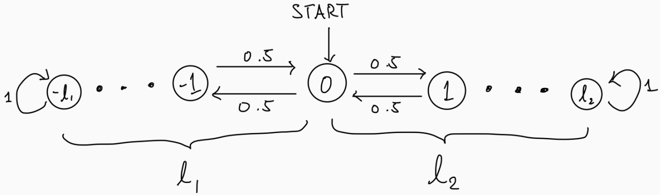

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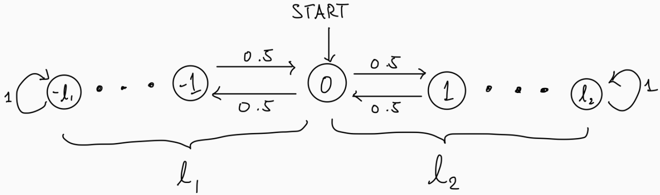

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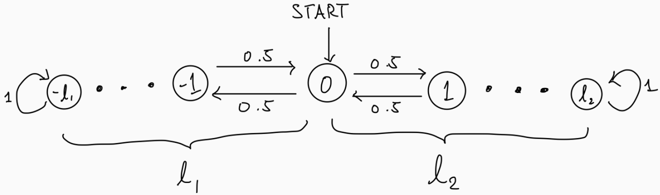

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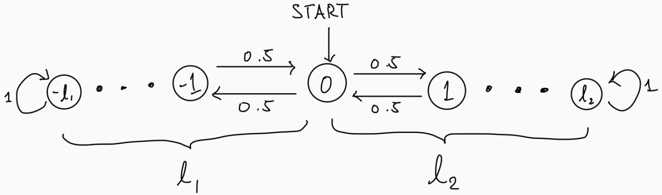

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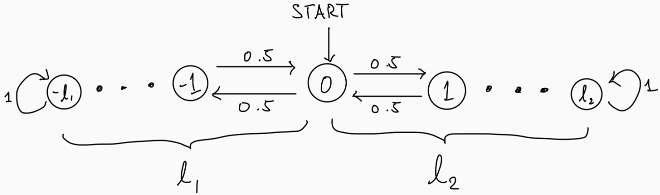

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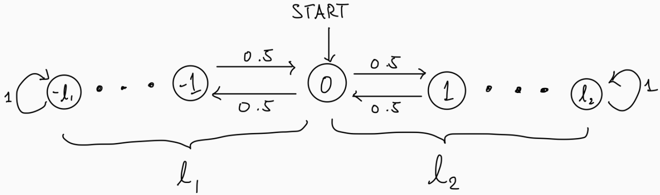

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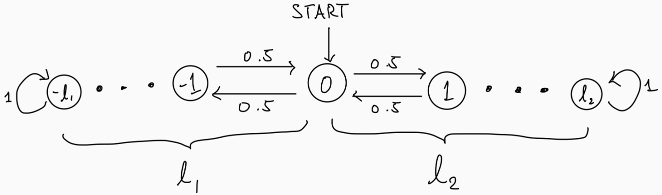

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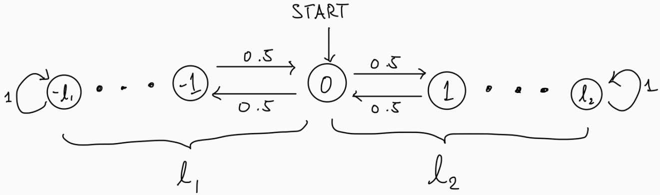

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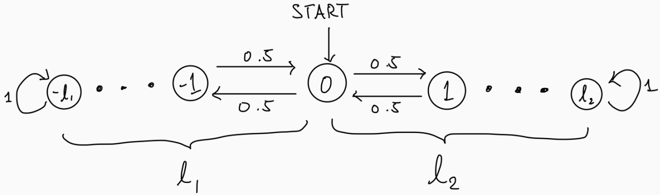
The Gambler's Ruin III



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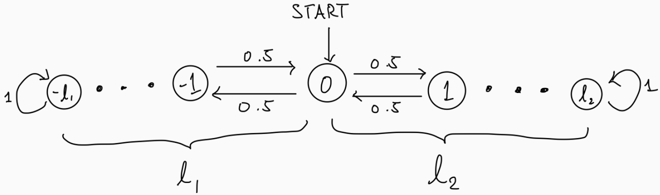

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Solve: $q = l_1/(l_1 + l_2)$. The more money you're willing to bet, the more you win!

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Is it irreducible? Yes, if it's connected.

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$$\pi_v = \sum_{u \in N(v)} \frac{d(u)}{2\,|E|} \frac{1}{d(u)} = \frac{d(v)}{2\,|E|}$$

.

So π solves the balance equations, so it's stationary.

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Since $v \in N(u)$: $h_{v,u} < 2\,|E|$

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So: $4|E||V|$ is an upper bound on the cover time.

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Find the limiting distribution by solving an eigenvalue problem! (Math 128B, Math 221)

Gig: Random Text