Alex Psomas: Lecture 20.

Chernoff and Erdős

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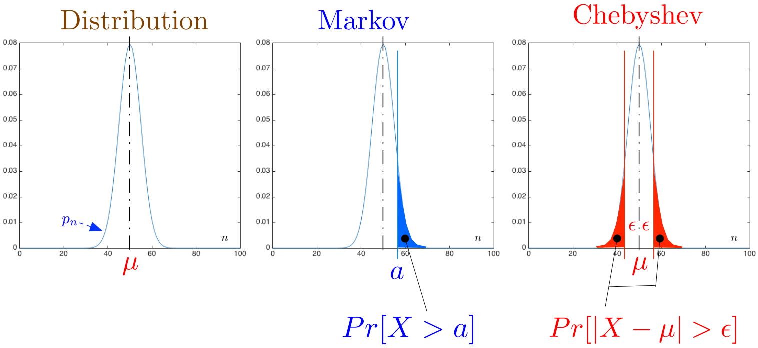
Chernoff and Erdős

1. Confidence intervals
2. Chernoff
3. Probabilistic Method

Reminders

- Quiz due tomorrow.
- Quiz coming out today.
- Midterm re-grade requests closing tomorrow.

Inequalities: An Overview



Confidence intervals example

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For $\varepsilon = 0.01$ we get that $n \geq 50000$ coins are sufficient.

Chernoff

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The ugly: People get scared the first time they see the bound.

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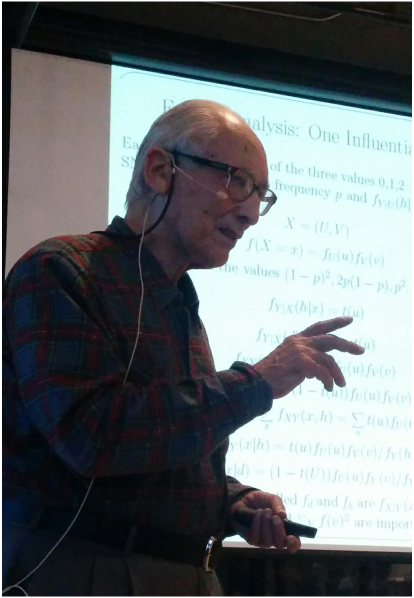
Herman Chernoff



With great proof comes great power

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Welcome to my life

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Yay!

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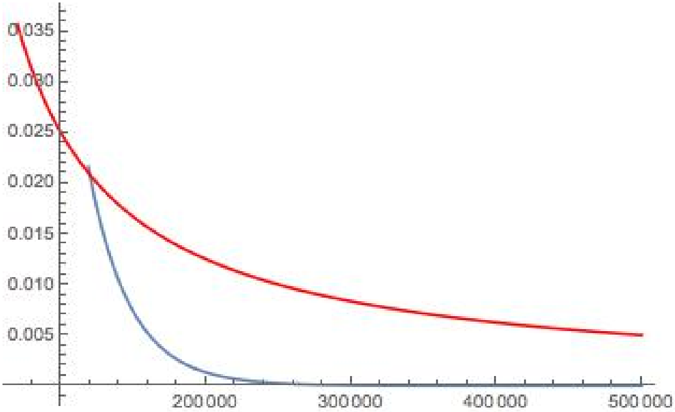

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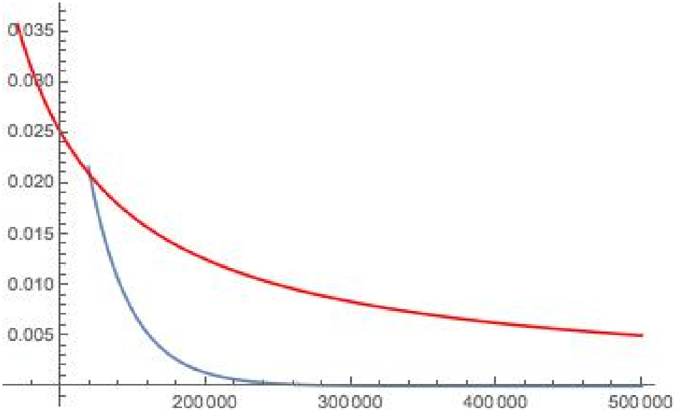

x axis is number of coins.

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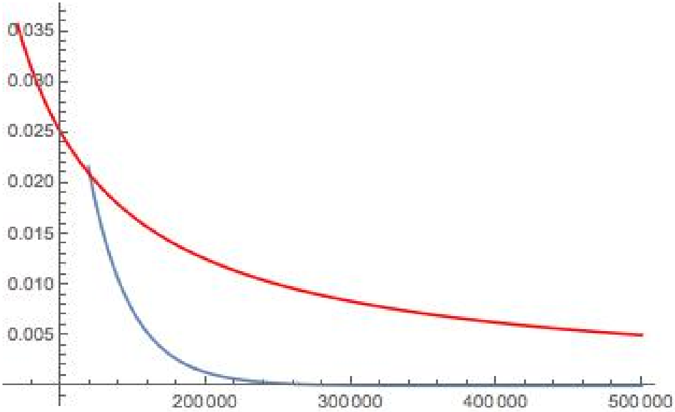

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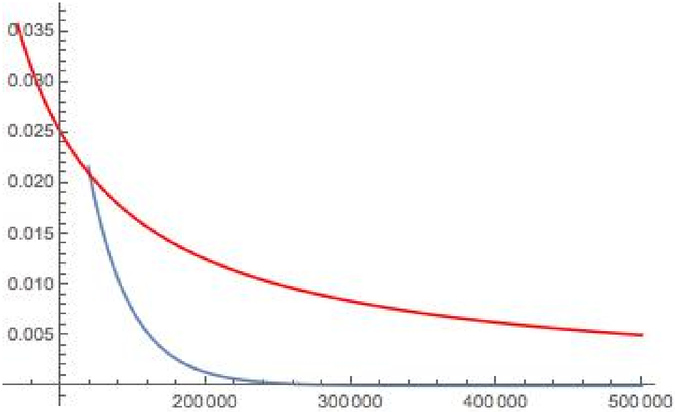

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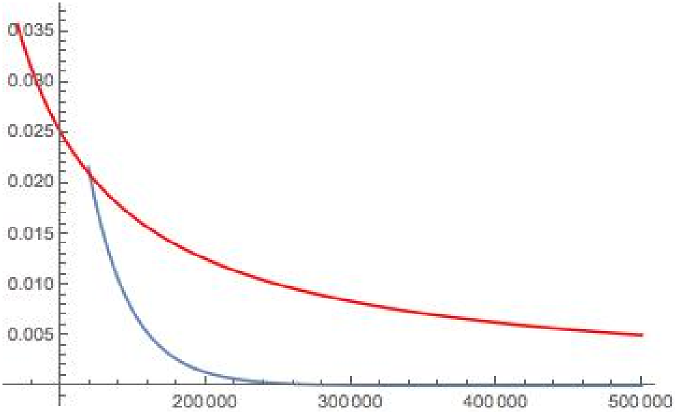

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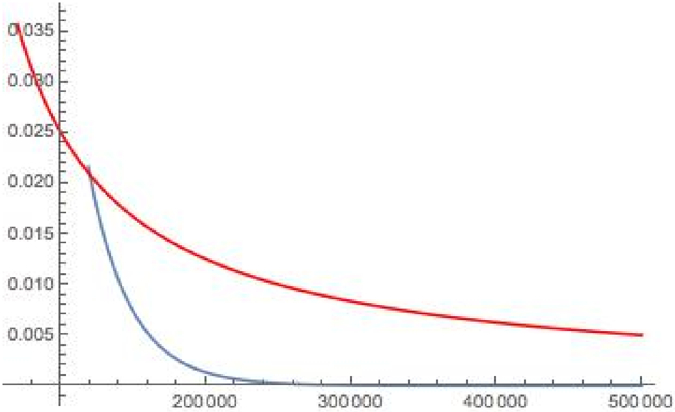

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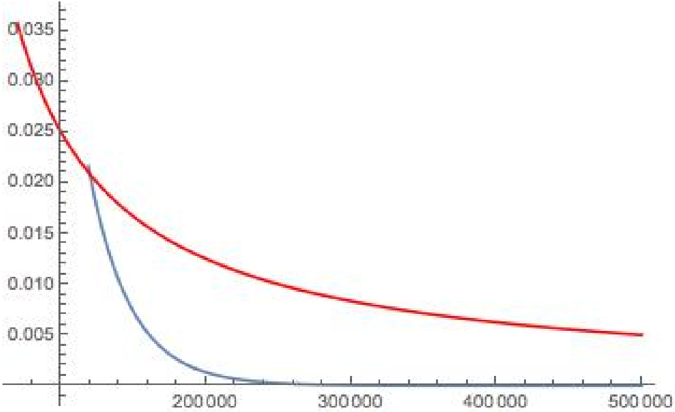

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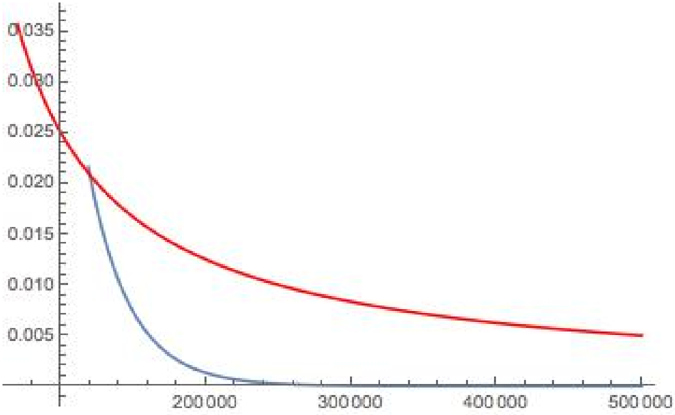

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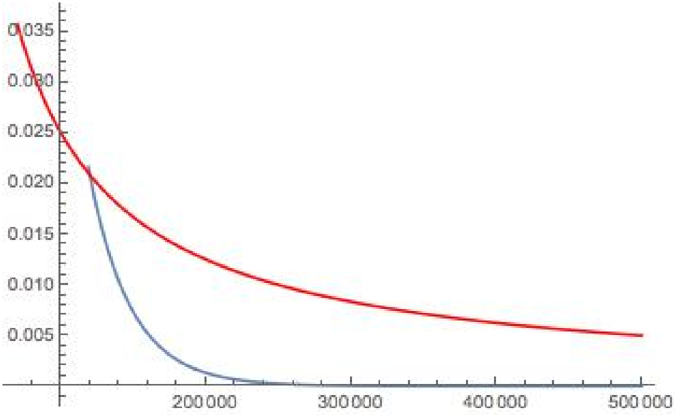

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For a million coins: Chebyshev: 0.0025

Chernoff: $3.33824 * 10^{-15}$

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6 volunteers

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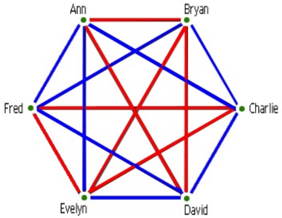

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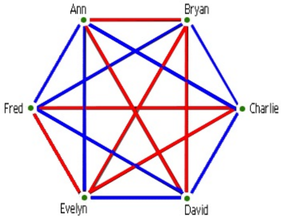

Blue edge if they know each other.

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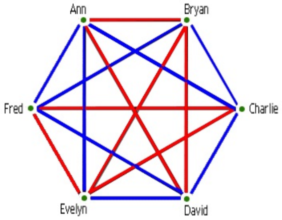

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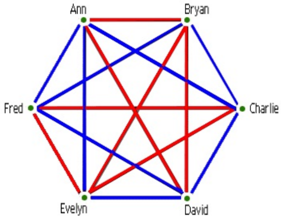
Red edge if they don’t know each other.

6 volunteers



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There is always a group of 3 that either all know each other, or all are strangers.

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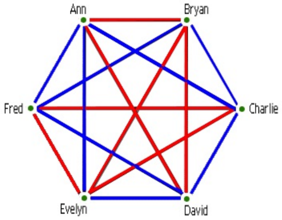

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There is always a group of 3 that either all know each other, or all are strangers.

There always exists a monochromatic triangle.

How can we show that things exist?

Say I have a group of 1000 people.

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Induction? Nah...

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The probabilistic method

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Well, that means that there is a coloring with no monochromatic clique of size k!

The probabilistic method

If I do something at random,

The probabilistic method

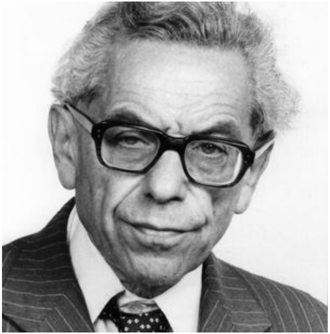
If I do something at random, and the probability I fail is strictly less than 1,

The probabilistic method

If I do something at random, and the probability I fail is strictly less than 1, that means that there is a way to succeed!!

The probabilistic method

Paul Erdős



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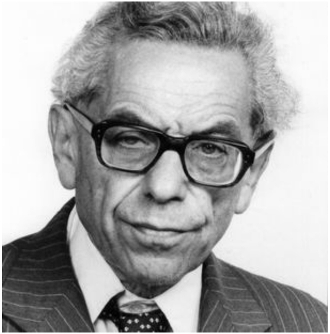

Many quotes:

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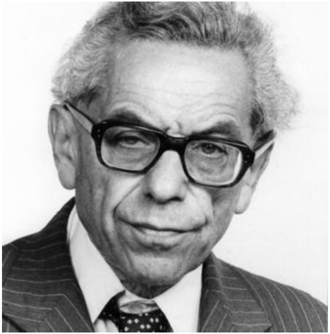

Many quotes:
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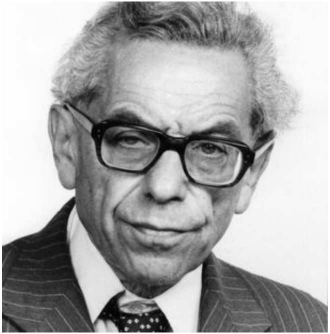

Many quotes:
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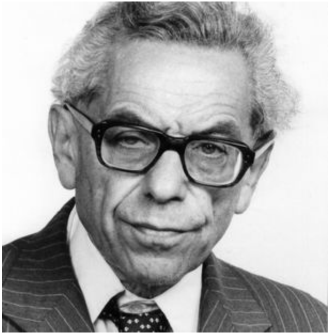

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It is not enough to be in the right place at the right time. You should also have an open mind at the right time.

Summary

Chernoff and Erdős

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Chernoff and Erdős

- Chernoff.
- The Probabilistic Method.