Alex Psomas: Lecture 18.

$\boxed{\text{Random Variables: Variance}}$

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1. Variance
2. Distributions

Variance

Flip a coin:

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Any other measures???

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Any other measures???
What else that's informative can we say?

Variance

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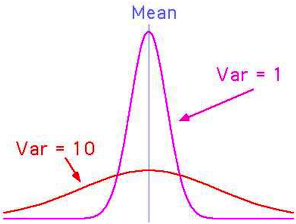

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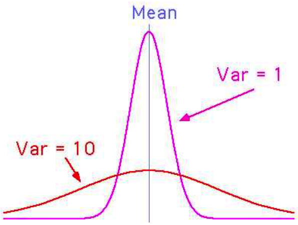

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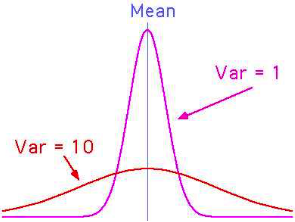
Variance



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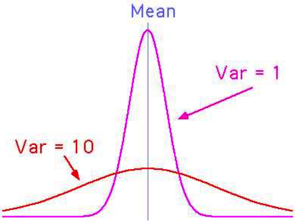

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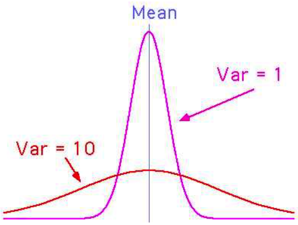

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$$\sigma^2(X) := var[X] = E[(X - E[X])^2].$$

$\sigma(X)$ is called the standard deviation of X.

Variance and Standard Deviation

Fact:

$$var[X] = E[X^2] - E[X]^2.$$

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Consider X with

$$X = \begin{cases} -1, & \text{w. p. } 0.99 \\ 99, & \text{w. p. } 0.01. \end{cases}$$

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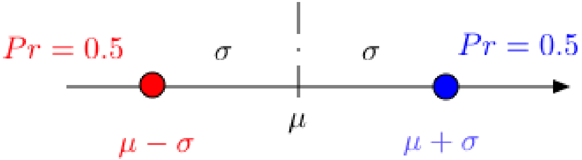

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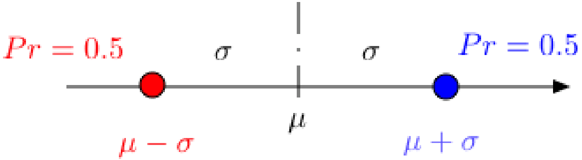

Consider the random variable X such that

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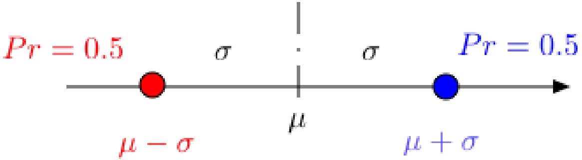

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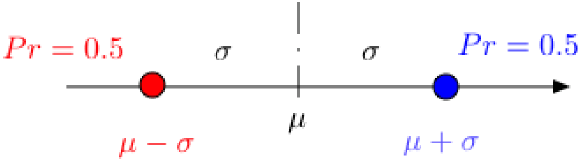

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Then, $E[X] = \mu$ and $E[(X - E[X])^2] = \sigma^2$. Hence,

$$var(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$

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$\square$

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$$\begin{aligned} var(X + Y + Z + \cdots) &= E((X + Y + Z + \cdots)^2) \\ &= E(X^2 + Y^2 + Z^2 + \cdots + 2XY + 2XZ + 2YZ + \cdots) \\ &= E(X^2) + E(Y^2) + E(Z^2) + \cdots + 0 + \cdots + 0 \\ &= var(X) + var(Y) + var(Z) + \cdots . \end{aligned}$$

$\square$

Distributions

- Bernoulli
- Binomial
- Uniform
- Geometric

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Flip a coin, with heads probability p.

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Jacob Bernoulli

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$$Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}, i = 0, 1, \ldots, n : B(n, p) \text{ distribution}$$

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Indicator for the i-th coin:

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$$E[X]=\sum_{m=1}^{n} mPr[X=m]=\sum_{m=1}^{n} m\times\frac{1}{n}=\frac{1}{n}\frac{n(n+1)}{2}=\frac{n+1}{2}.$$

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This gives

$$var(X) = \frac{1+3n+2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}.$$

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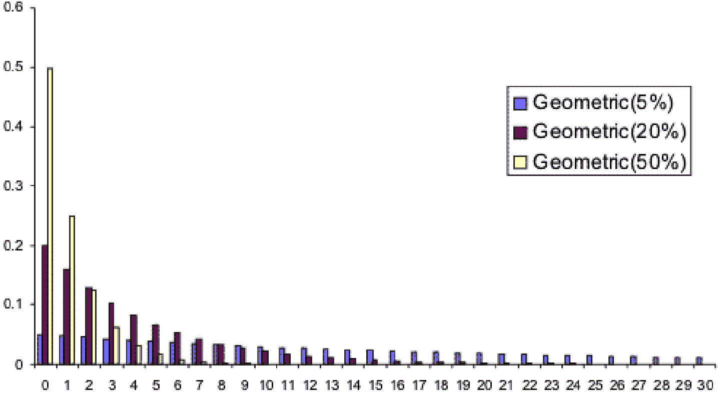

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Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[$"get second distinct coupon"$|$"got first coupon$''] = \frac{n-1}{n}$

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Review: Harmonic sum

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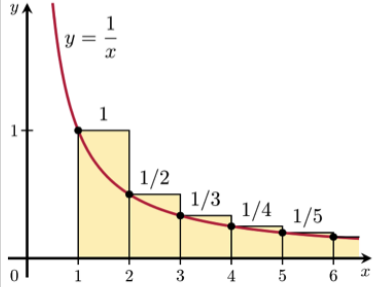

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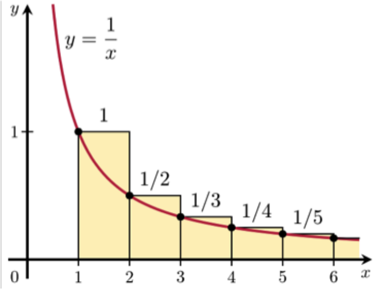

A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

Harmonic sum: Paradox

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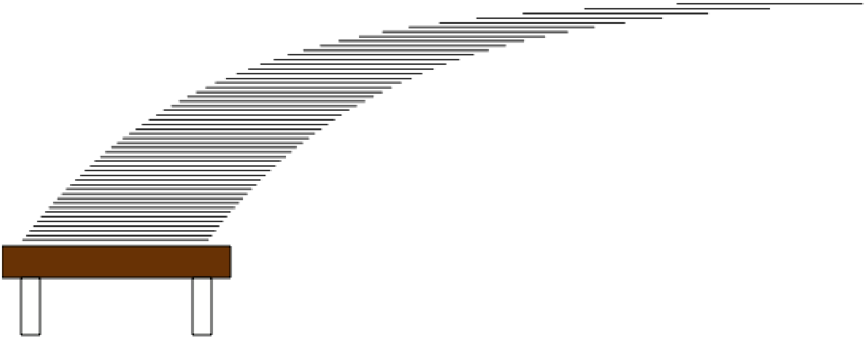

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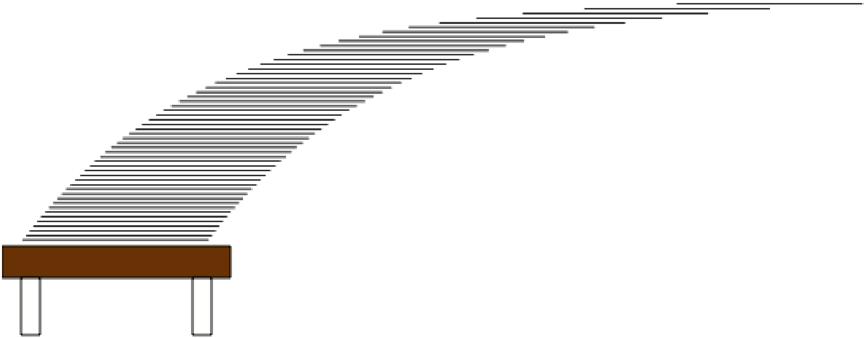

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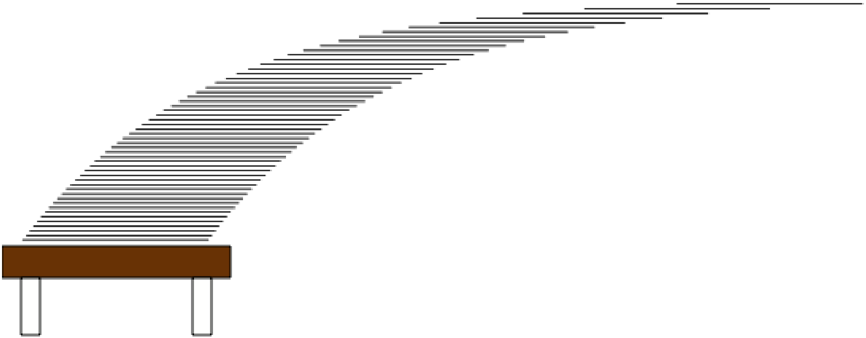
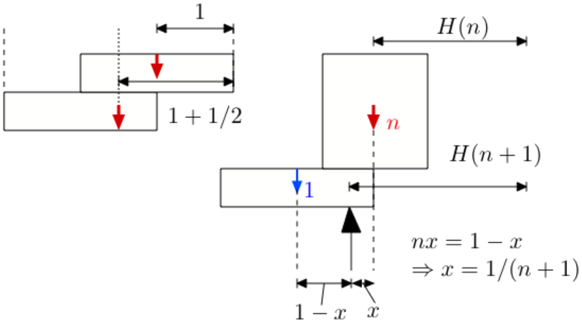


If each card has length 2, the stack can extend $H(n)$ to the right of the table. As n increases, you can go as far as you want!

Stacking



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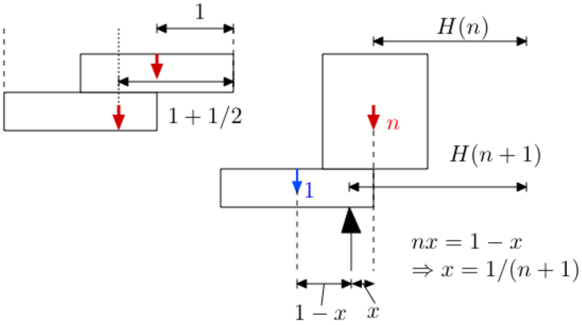

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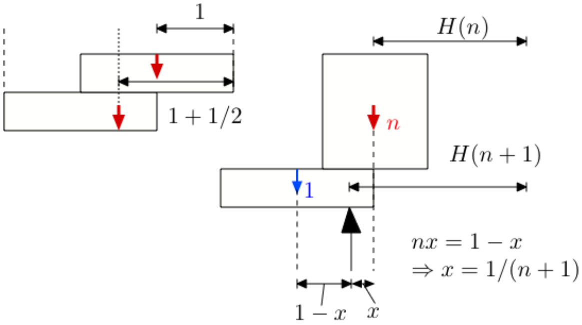

The cards have width 2. Induction shows that the center of gravity after n cards is $H(n)$ away from the right-most edge.

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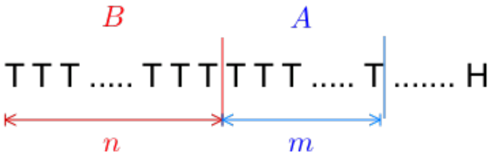

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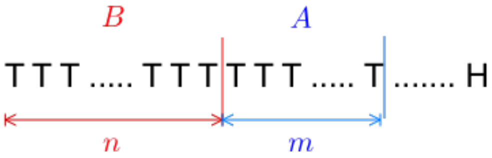

The coin is memoryless, therefore, so is X.

Geometric Distribution: Yet another look

Theorem: For a r.v. X that takes the values $\{0,1,2,\ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]

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$\square$

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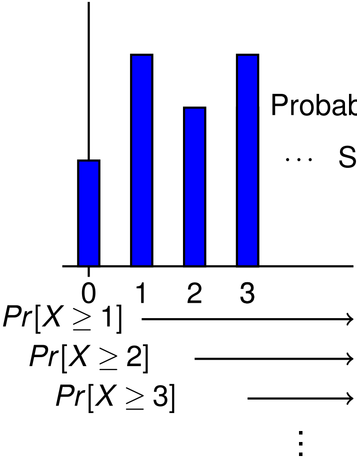

Probability mass at i, counted i times.

$\cdots$ Same as $\sum_{i=1}^{\infty} i \times Pr[X = i]$.

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$\boxed{\text{Random Variables}}$

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