Alex Psomas: Lecture 17.

Random Variables: Expectation, Variance

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1. Random Variables, Expectation: Brief Review
2. Independent Random Variables.
3. Variance

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$$Pr[X = 2] = Pr[X^{-1}(2)] = Pr[\{HHT, HTH, THH\}]$$

$$= Pr[\{HHT\}] + Pr[\{HTH\}] + Pr[\{THH\}] = \frac{3}{8}$$

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- $X \cos(2\pi Y + Z)$.

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$$E[X] = \sum_\omega X(\omega) \times Pr[\omega].$$

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$$\sum_{\omega} X(\omega) Pr[\omega] = 3\frac{1}{8} + 2\frac{1}{8} + 2\frac{1}{8} + 2\frac{1}{8} + 1\frac{1}{8} + 1\frac{1}{8} + 1\frac{1}{8} + 0\frac{1}{8}.$$

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Also,

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The fact that this average converges to $E[X]$ is a theorem:

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An Illustration: Rolling Dice

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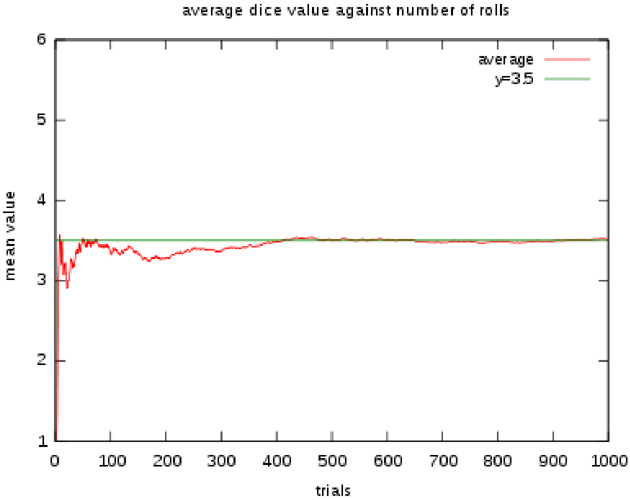

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$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

is called the indicator of the event A.

Note that $Pr[X = 1] = Pr[A]$ and $Pr[X = 0] = 1 - Pr[A]$.

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Thus, we will write $X = 1_A$.

Linearity of Expectation

Theorem:

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&\quad= \sum_{\omega}(a_1X_1(\omega)+\cdots+a_nX_n(\omega))Pr[\omega] \\
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Note: If we had defined $Y=a_1X_1+\cdots+a_nX_n$ has had tried to compute $E[Y]=\sum_y yPr[Y=y]$, we would have been in trouble!

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Note: Computing $\sum_x xPr[X = x]$ directly is not easy!

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For a group of 28 it's about 1. For 100 it's 13.5. For 280 it's 107.

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Thus,

$$E[Y] = 4\frac{2}{6} + 1\frac{2}{6} + 0\frac{1}{6} + 9\frac{1}{6} = \frac{19}{6}.$$

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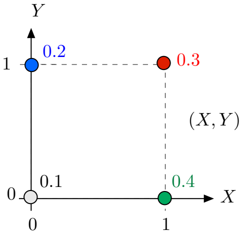

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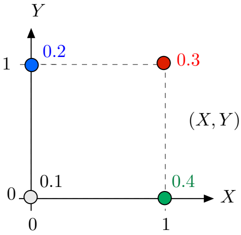

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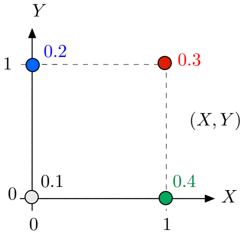

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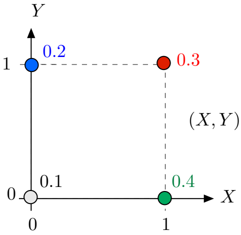

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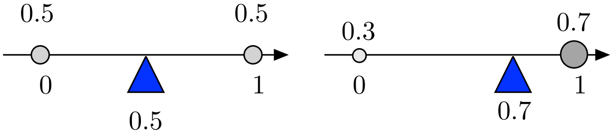
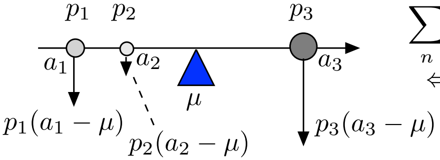
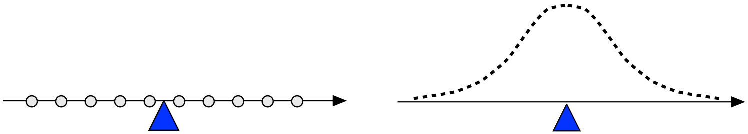




$$\sum_n p_n(a_n - \mu) = 0$$

$$\Leftrightarrow \mu = \sum_n a_n p_n = E[X]$$



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Unfortunately, we won't talk about this in this class...

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Functions of Independent random Variables

Theorem Functions of independent RVs are independent
Let X, Y be independent RV. Then

$$f(X) \text{ and } g(Y) \text{ are independent, for all } f(\cdot), g(\cdot).$$

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E[(X-Y)^2] &= E[X^2+Y^2-2XY] = 2E[X^2]-2E[X]^2 \\
&= \frac{1+3n+2n^2}{3} - \frac{(n+1)^2}{2}.
\end{aligned}
$$

Mutually Independent Random Variables

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X, Y, Z are mutually independent if

$$Pr[X = x, Y = y, Z = z] = Pr[X = x]Pr[Y = y]Pr[Z = z], \text{ for all } x, y, z.$$

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$$Pr[1_A = 1, 1_B = 1, 1_C = 1] = Pr[A \cap B \cap C], \ldots$$

$\square$

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Any other measures???

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Any other measures??? What else that's informative can we say?

Variance

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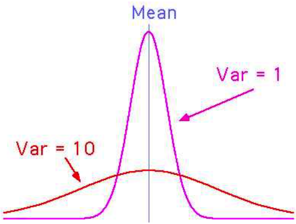
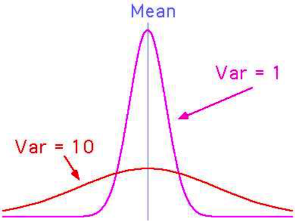


Variance



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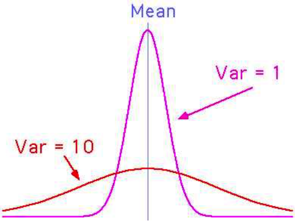

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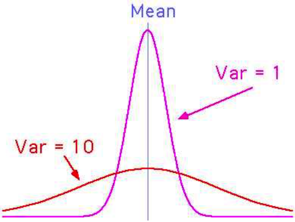

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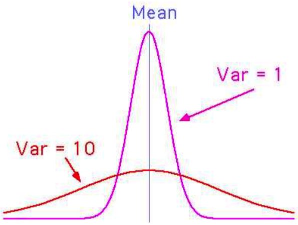

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$\sigma(X)$ is called the standard deviation of X.

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Fact:

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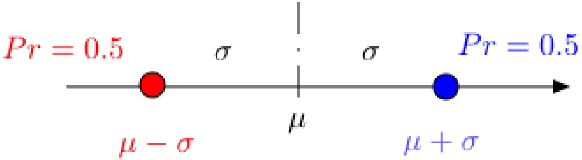

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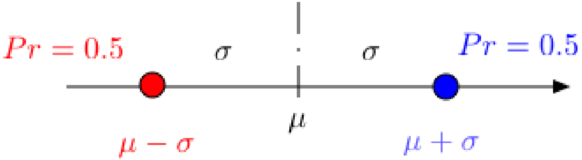

Consider the random variable X such that

$$X = \begin{cases} \mu - \sigma, & \text{w.p. } 1/2 \\ \mu + \sigma, & \text{w.p. } 1/2. \end{cases}$$

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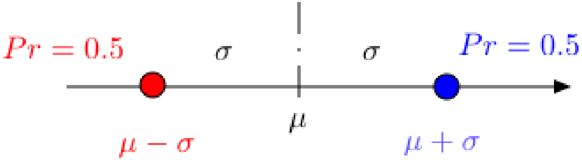

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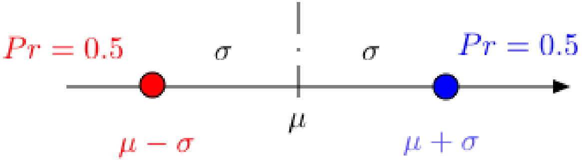

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$$var(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$

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Consider X with

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Proof:

$$Var(cX) = E((cX)^2) - (E(cX))^2$$

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Age group	18–24		25–34		35–44		45–54		55–54	
Smoker	Y	N	Y	N	Y	N	Y	N	Y	N
Dead	2	1	3	5	11	7	27	12	51	40
Alive	53	61	121	152	95	114	103	66	64	81
Ratio	2.3		0.75		2.4		1.44		1.61	

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In each separate category, the percentage of fatalities among smokers is higher, and yet the overall percentage of fatalities among smokers is lower!

Summary

Random Variables

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- A random variable X is a function $X : \Omega \to \Re$.
- $Pr[X = a] := Pr[X^{-1}(a)] = Pr[\{\omega \mid X(\omega) = a\}]$.
- $Pr[X \in A] := Pr[X^{-1}(A)]$.
- The distribution of X is the list of possible values and their probability: $\{(a, Pr[X = a]), a \in \mathscr{A}\}$.
- $g(X, Y, Z)$ assigns the value .... .
- $E[X] := \sum_a aPr[X = a]$.
- Expectation is Linear.
- Independent Random Variables.
- Variance.