Alex Psomas: Lecture 14.

Events, Conditional Probability, Independence, Bayes' Rule

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1. Probability Basics Review
2. Conditional Probability
3. Independence of Events
4. Bayes' Rule

Probability Basics Review

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- Random Experiment.

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  Flip a fair coin twice.

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See next two slides for (a) and (c).

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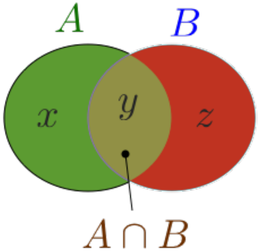

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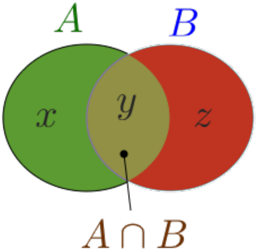

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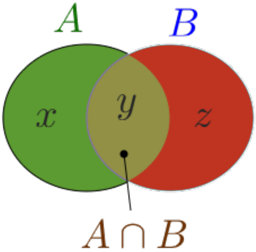

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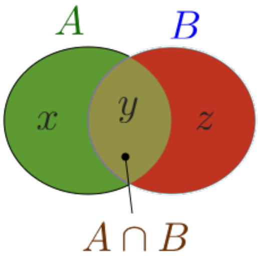

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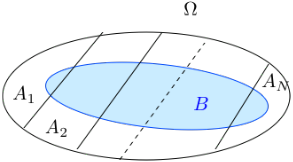
Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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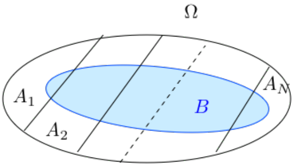

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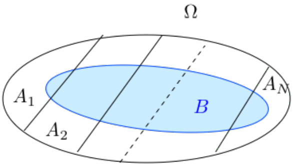

Then,

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Indeed, B is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$.

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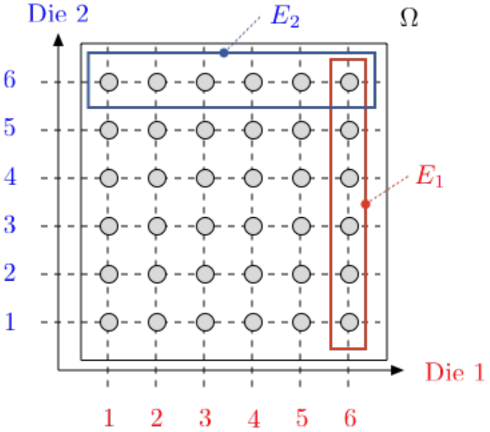

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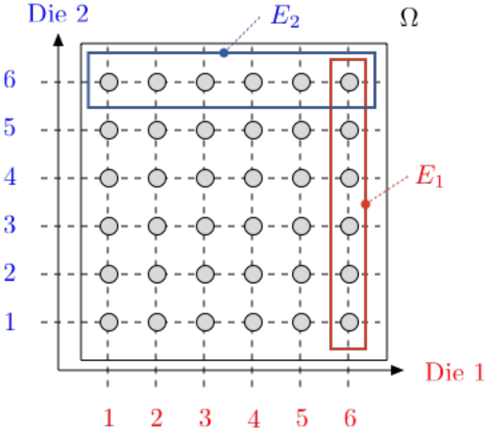

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$E_1 =$ 'Red die shows 6';

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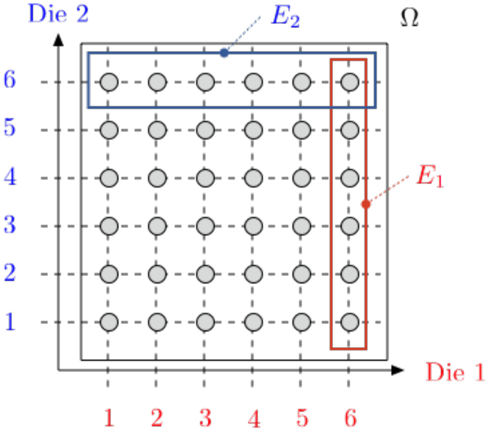

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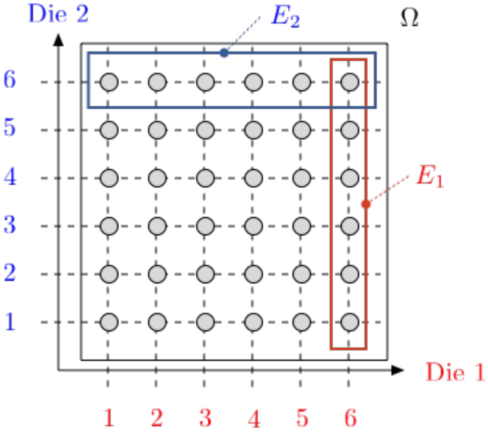

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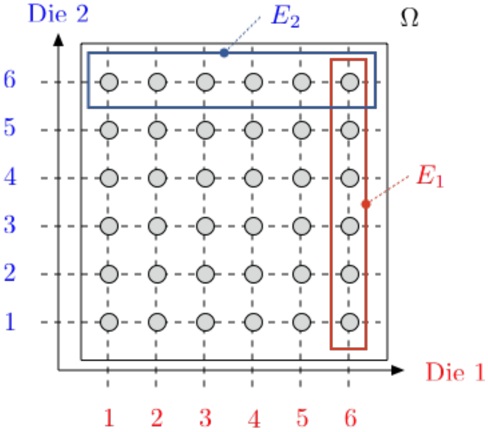

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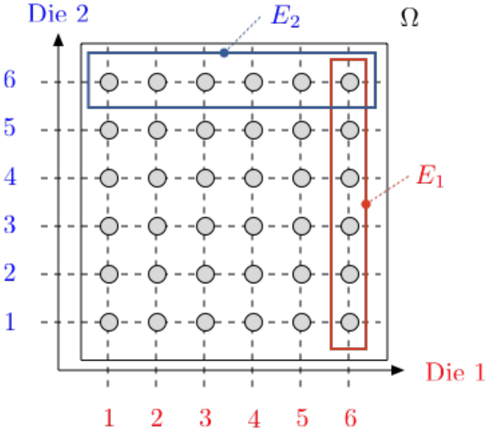

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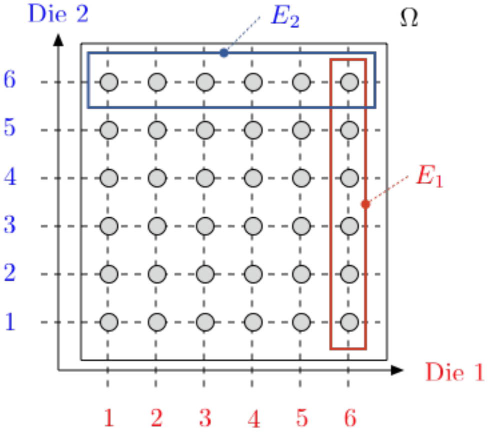

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$$Pr[E_1] = \frac{6}{36}, Pr[E_2] = \frac{6}{36}, Pr[E_1 \cup E_2] = \frac{11}{36}.$$

Conditional probability: example.

Two coin flips (fair coin).

Conditional probability: example.

Two coin flips (fair coin). First flip is heads.

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Two coin flips (fair coin). First flip is heads. Probability of two heads?

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Two coin flips (fair coin). First flip is heads. Probability of two heads?
$\Omega = \{HH, HT, TH, TT\}$;

Conditional probability: example.

Two coin flips (fair coin). First flip is heads. Probability of two heads?

$\Omega = \{HH, HT, TH, TT\}$; Uniform probability space.

Conditional probability: example.

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Event $A =$ first flip is heads:

Conditional probability: example.

Two coin flips (fair coin). First flip is heads. Probability of two heads?

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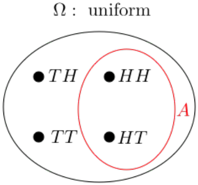
Event A = first flip is heads: $A = \{HH, HT\}$.



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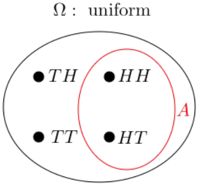

New sample space: A;

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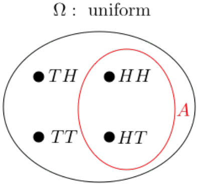

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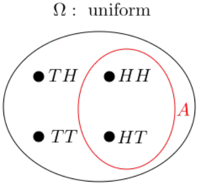

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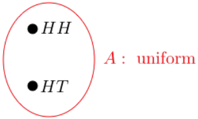

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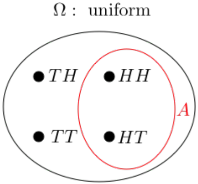

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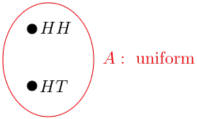

Event $B =$ two heads.

Conditional probability: example.

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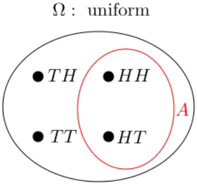

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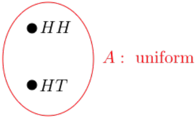

Event $B =$ two heads.

The probability of two heads if the first flip is heads.

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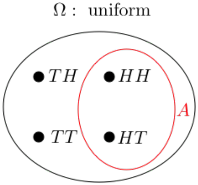

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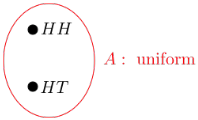

Event $B =$ two heads.

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The probability of B given A

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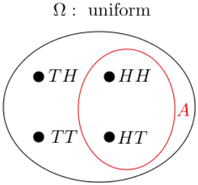

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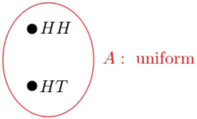

Event $B =$ two heads.

The probability of two heads if the first flip is heads.

The probability of B given A is $1/2$.

A similar example.

Two coin flips(fair coin).

A similar example.

Two coin flips(fair coin). At least one of the flips is heads.

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Two coin flips(fair coin). At least one of the flips is heads.
$\rightarrow$ Probability of two heads?

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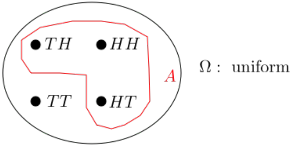

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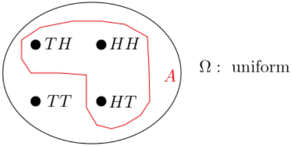

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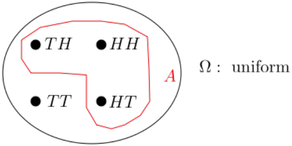

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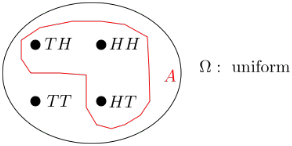

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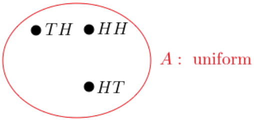

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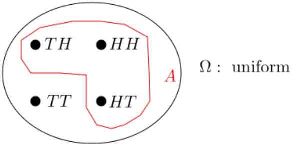

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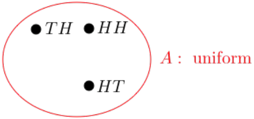

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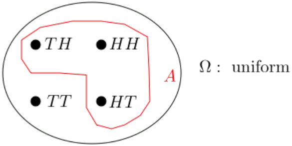

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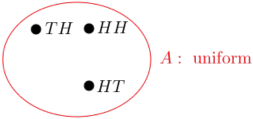

Event $B =$ two heads.

The probability of two heads if at least one flip is heads.

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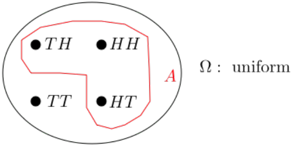

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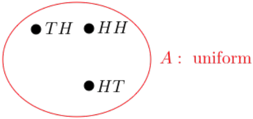

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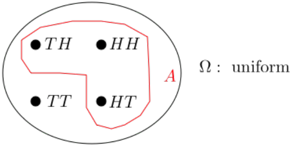

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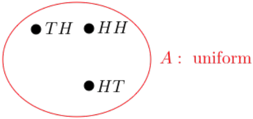

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Conditional Probability: A non-uniform example

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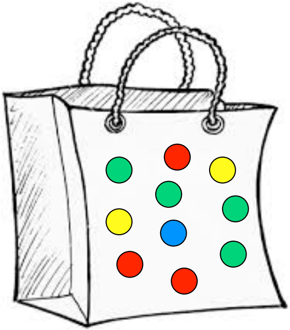


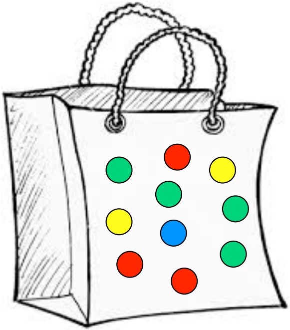
Physical experiment

Ω

	$Pr[\omega]$
Red	3/10
Green	4/10
Yellow	2/10
Blue	1/10

Probability model

Conditional Probability: A non-uniform example



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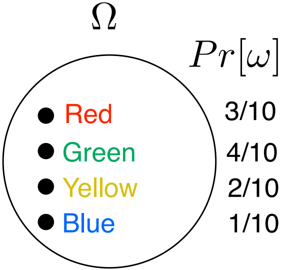

Probability model

$\Omega = \{$Red, Green, Yellow, Blue$\}$

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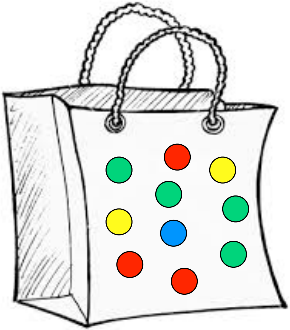

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$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] =$$

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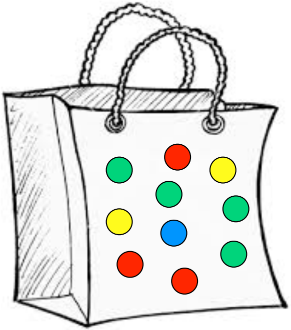

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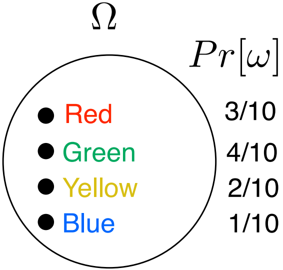
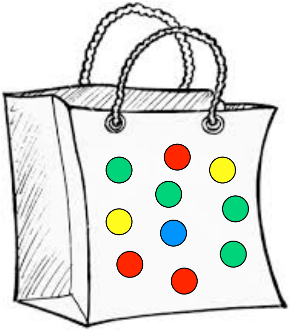


Probability model

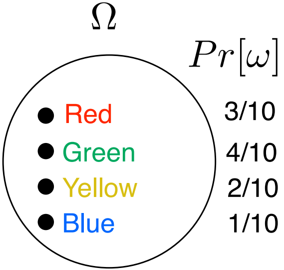
$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{3}{7} =$$

Conditional Probability: A non-uniform example



Physical experiment



Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{3}{7} = \frac{Pr[\text{Red} \cap (\text{Red or Green})]}{Pr[\text{Red or Green}]}$$

Another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

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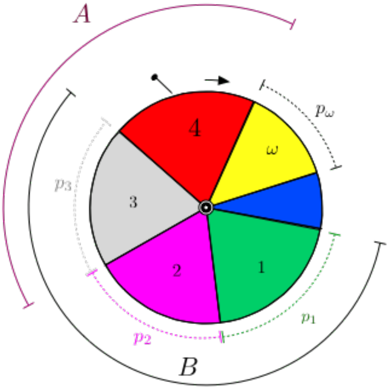

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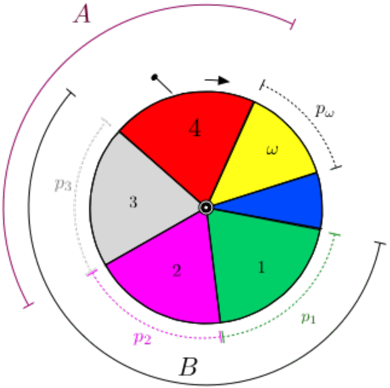

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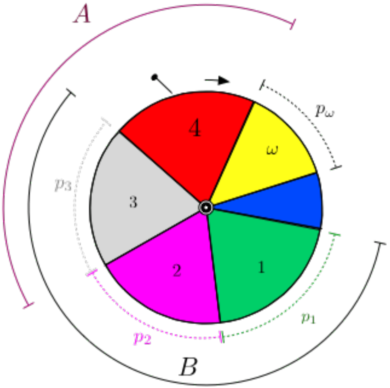

$$Pr[A|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

Yet another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

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Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.
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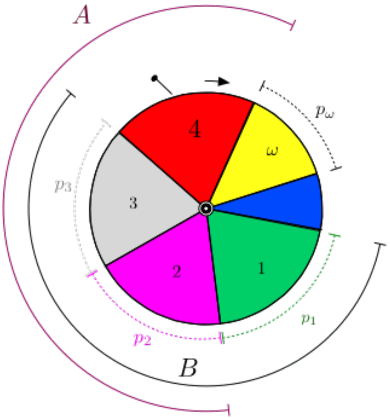

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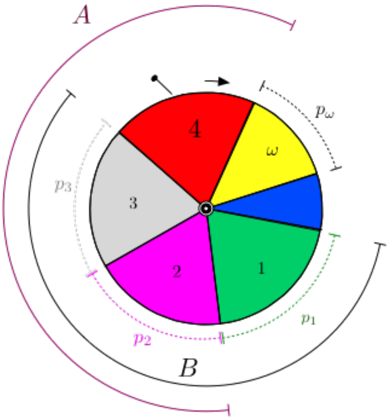

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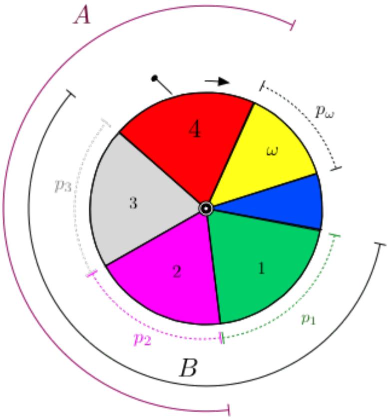
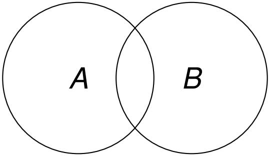


$$Pr[A|B] = \frac{p_2 + p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

Conditional Probability.

Definition: The **conditional probability** of B given A is

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$



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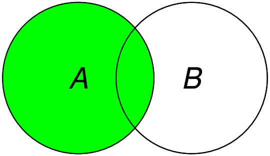

In A!

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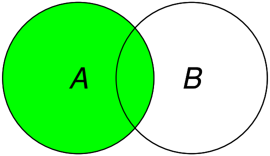

In A!
In B?

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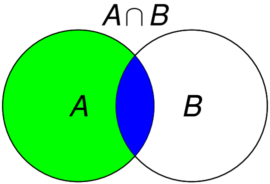

In A!
In B?
Must be in $A \cap B$.

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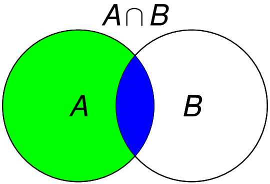

In A!

In B?

Must be in $A \cap B$.

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}.$$

More fun with conditional probability.

Toss a red and a blue die, sum is 4,

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Toss a red and a blue die, sum is 4,
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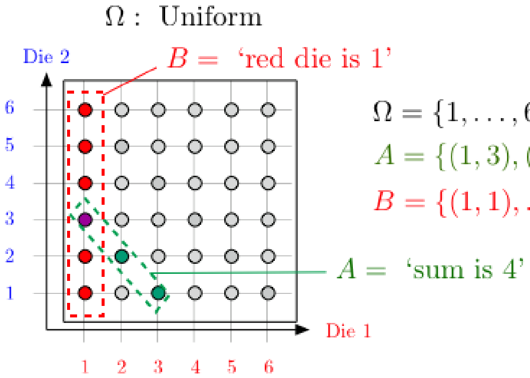

$\Omega = \{1, \ldots, 6\}^2$

$A = \{(1,3),(2,2),(3,1)\}$

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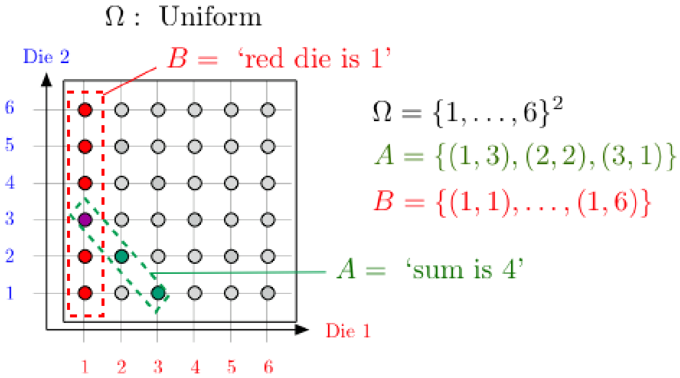

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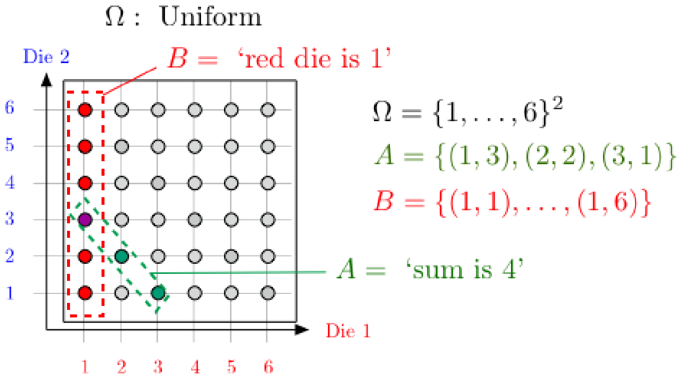

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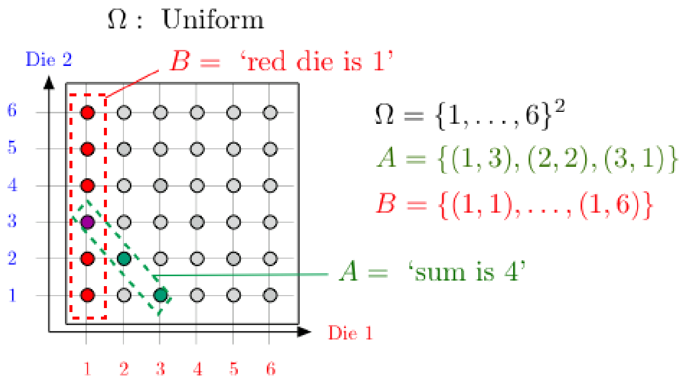

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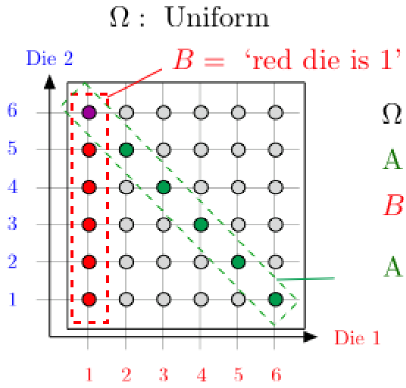

$\Omega = \{1, \ldots, 6\}^2$

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A = 'sum is 7'

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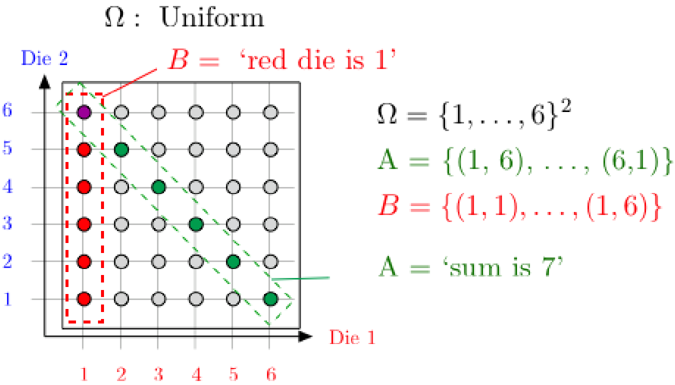

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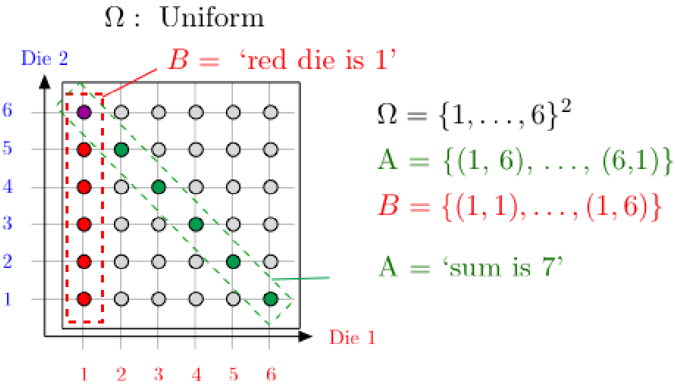

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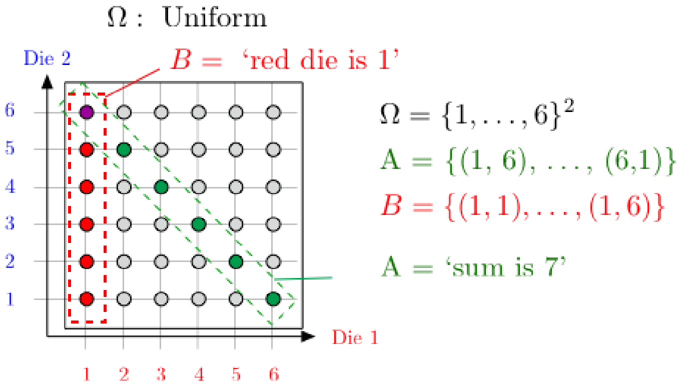

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Observing *A* does not change your mind about the likelihood of *B*.

Emptiness..

Suppose I toss 3 balls into 3 bins.

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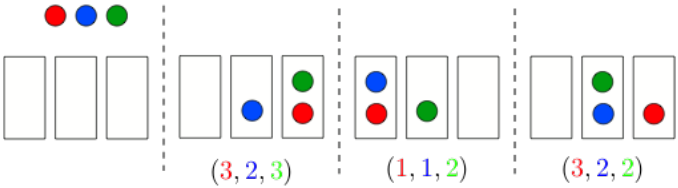

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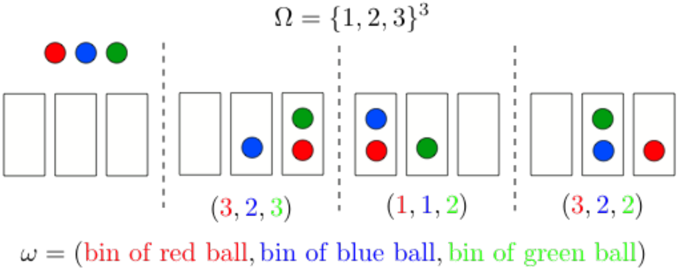

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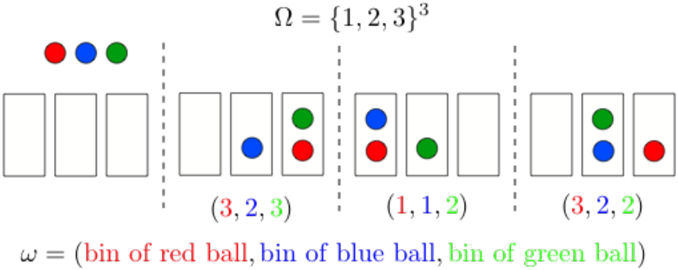

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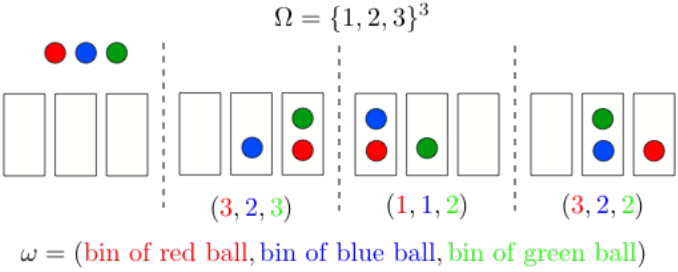

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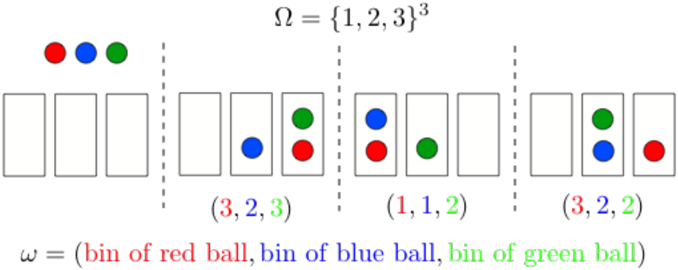

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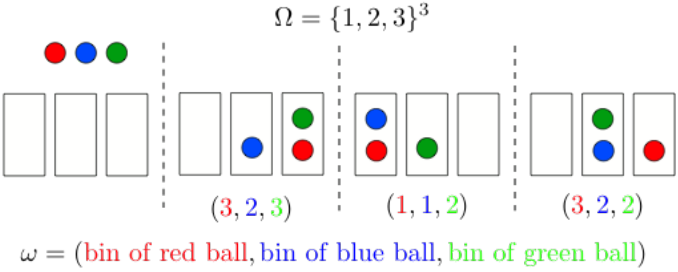

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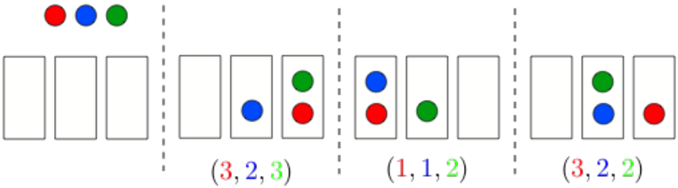

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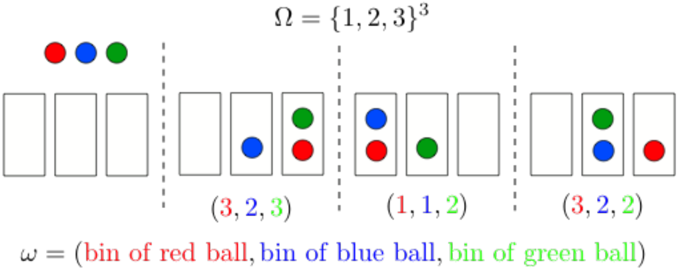

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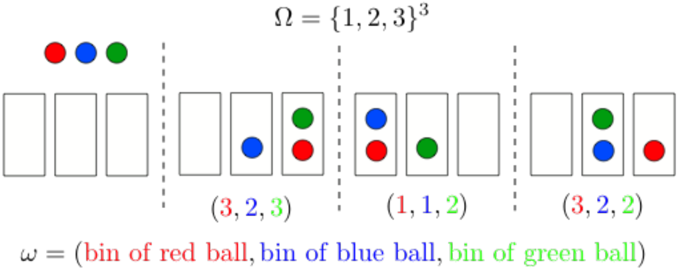

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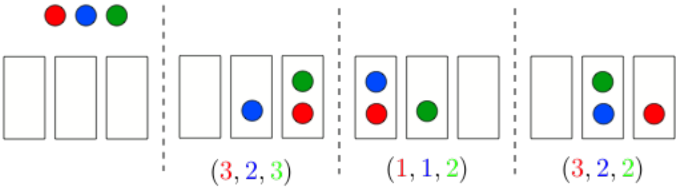

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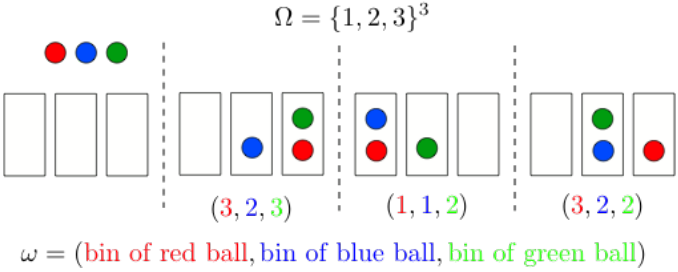

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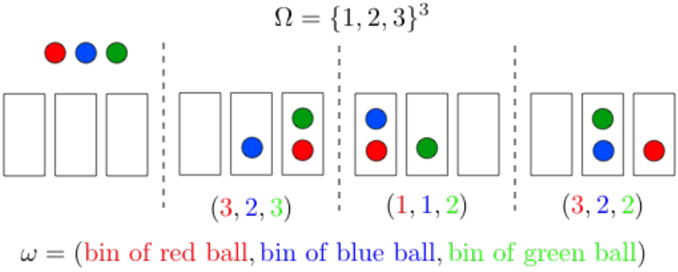

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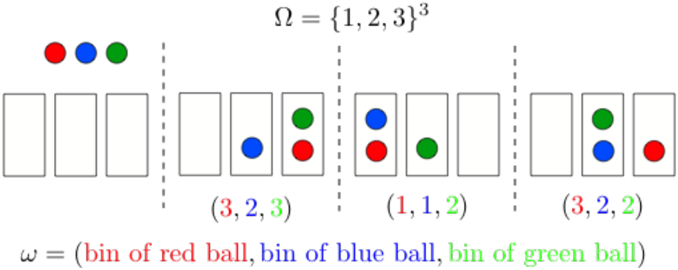

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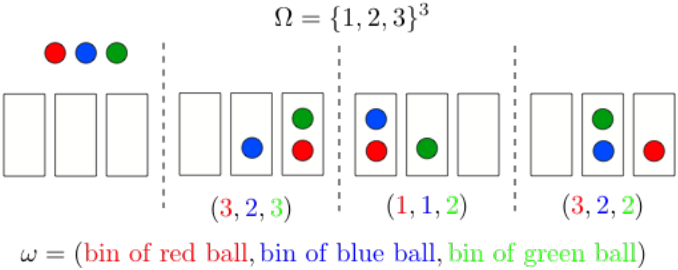

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A is less likely given B: If second bin is empty the first is more likely to have balls in it.

Gambler's fallacy.

Flip a fair coin 51 times.

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The likelihood of 51st heads does not depend on the previous flips.

Product Rule

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so that the result holds for $n+1$. $\square$

Correlation

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Random experiment: Pick a person at random.

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Event A: the person has lung cancer.

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- Smoking increases the probability of lung cancer by 17%.

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Event A: the person has lung cancer.
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Conclusion:

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- Smoking causes lung cancer.

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Event A: the person has lung cancer. Event B: the person is a heavy smoker. $Pr[A|B] = 1.17 \times Pr[A]$.

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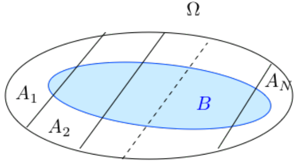
Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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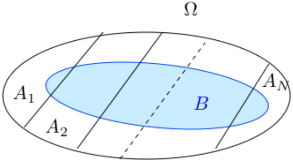

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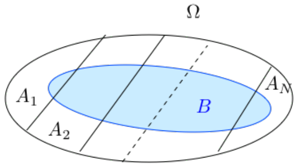

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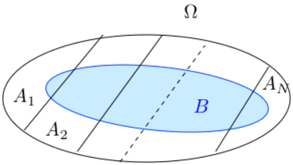

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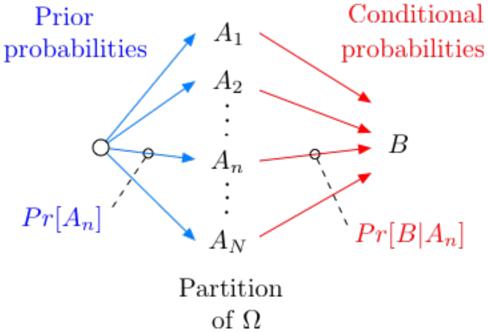

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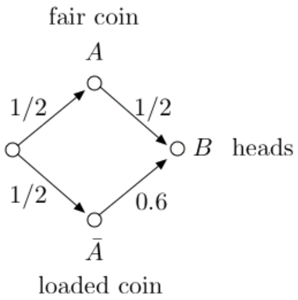

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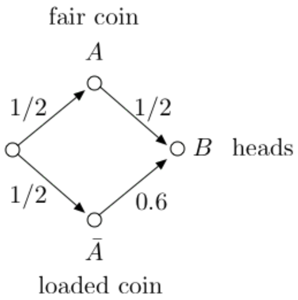

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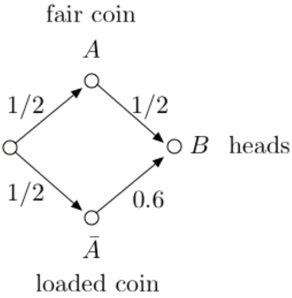

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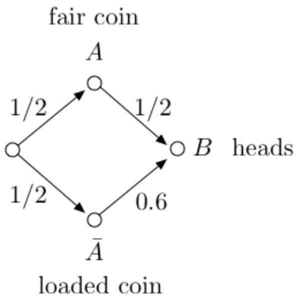

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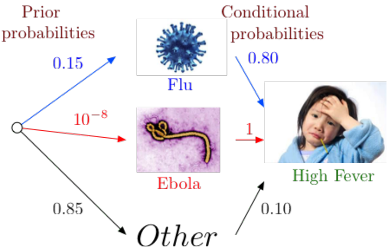

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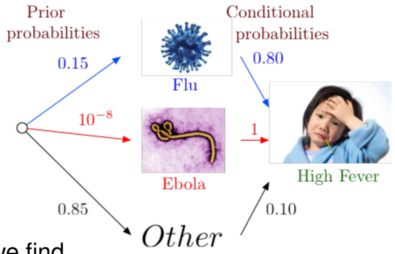

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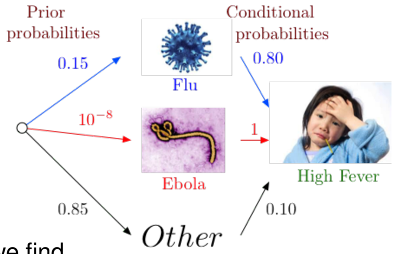

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$$Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58$$

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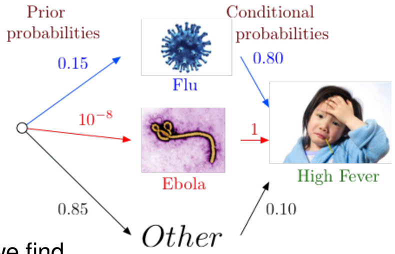

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$$Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58$$

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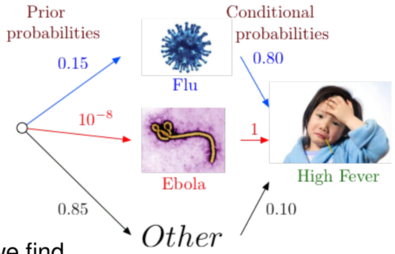
Why do you have a fever?



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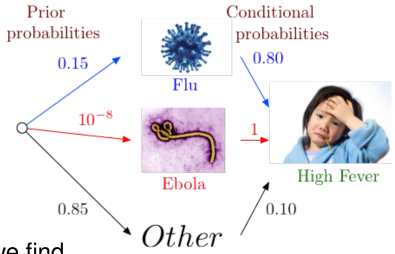

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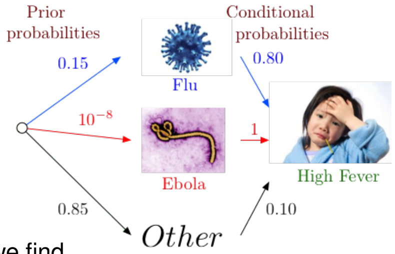

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These are the posterior probabilities. One says that 'Flu' is the Most Likely a Posteriori (MAP) cause of the high fever.

Bayes' Rule Operations

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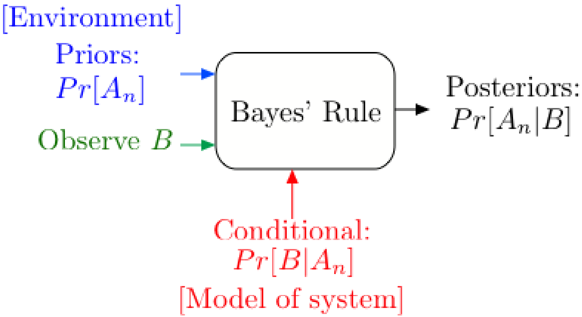

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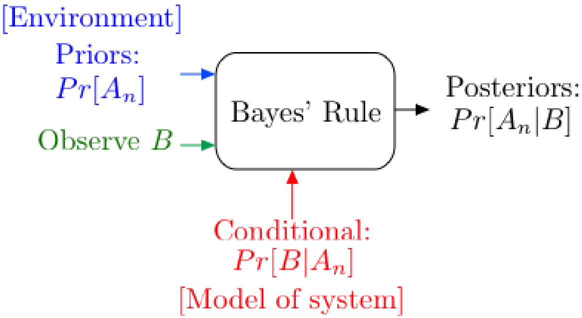

Bayes' Rule is the canonical example of how information changes our opinions.

Thomas Bayes

Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born	c. 1701 London, England
Died	7 April 1761 (aged 59) Tunbridge Wells, Kent, England
Residence	Tunbridge Wells, Kent, England
Nationality	English
Known for	Bayes' theorem

Source: Wikipedia.

Thomas Bayes

A

FIG. 3. *Joshua Bayes (1671–1746).*

$\xrightarrow{Pr[B|A]}$

Max over B

B

Thomas Bayes?

A Bayesian picture of Thomas Bayes.

Testing for disease.

Let's watch TV!!

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Random Experiment: Pick a random male.

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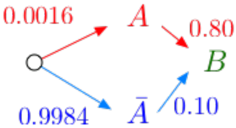
$$Pr[A|B]???$$

Bayes Rule.



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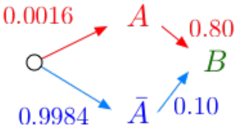

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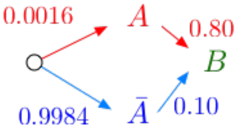

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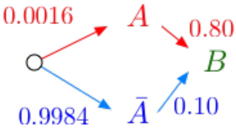

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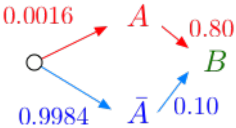

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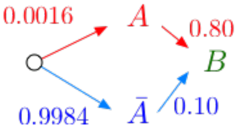

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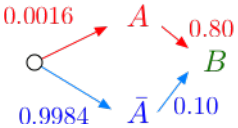

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Monty Hall.

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- All these are possible:

  $Pr[A|B] < Pr[A]; Pr[A|B] > Pr[A]; Pr[A|B] = Pr[A]$.