CS70: Alex Psomas: Lecture 13.

Modeling Uncertainty: Probability Space

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Modeling Uncertainty: Probability Space

1. Key Points
2. Random Experiments
3. Probability Space
4. Events

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- Probability
  - Models knowledge about uncertainty
  - Discovers best way to use that knowledge in making decisions

The Magic of Probability

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Uncertainty:

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Uncertainty: vague,

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Uncertainty: vague, fuzzy,

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Your cost: focused attention and practice on examples and problems.

A cool trick

Random Experiment: Flip one Fair Coin

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  (*One flip yields either ‘heads’ or ‘tails’.*)

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- Likelihoods: $H : 50\%$ and $T : 50\%$

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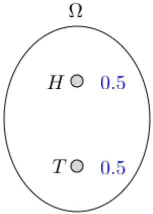
Physical Experiment



Probability Model

Random Experiment: Flip one Fair Coin

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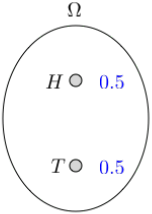

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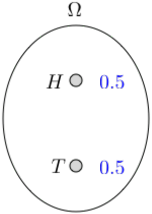

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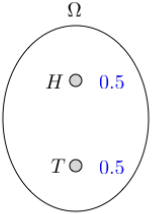

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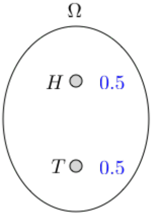

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  - A set Ω of outcomes: $\Omega = \{H, T\}$.

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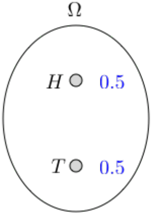

Probability Model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)
- The Probability model is simple:
  - A set Ω of outcomes: $\Omega = \{H, T\}$.
  - A probability assigned to each outcome: $Pr[H] = 0.5, Pr[T] = 0.5$.

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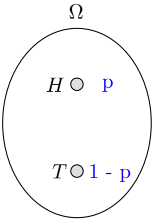

Probability Model

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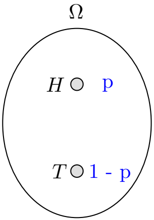

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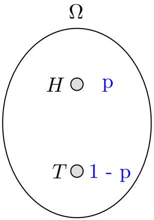

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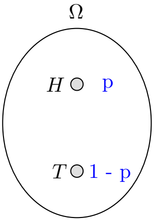

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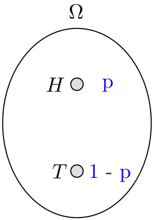

Probability Model

- Same set of outcomes as before!
- Different probabilities!
- The most common mistake in Probability: assuming that outcomes are equally likely.

Flip Two Fair Coins

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- Possible outcomes:

Flip Two Fair Coins

- Possible outcomes: $\{HH, HT, TH, TT\}$

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Flip Glued Coins

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Flips two coins glued together side by side:

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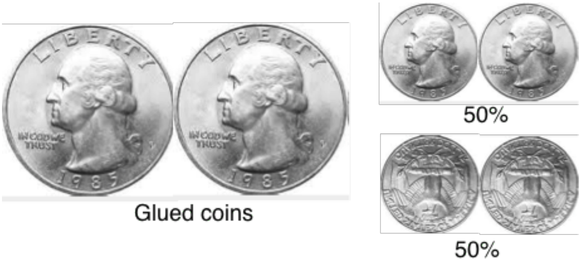

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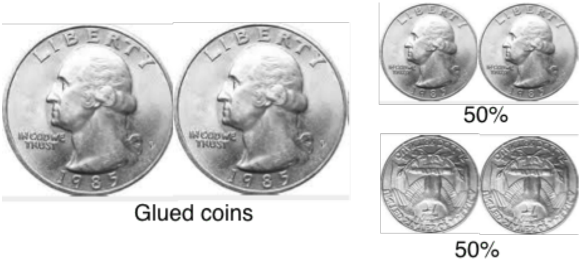

- Possible outcomes: $\{HH, TT\}$.
- Likelihoods: $HH : 0.5, TT : 0.5$.
- Note: Coins are glued so that they show the same face.

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Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: $\{HT, TH\}$.
- Likelihoods: $HT : 0.5, TH : 0.5$.
- Note: Coins are glued so that they show different faces.

Flip two Attached Coins

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Flips two coins attached by a spring:

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Flips two coins attached by a spring:

- Possible outcomes: $\{HH, HT, TH, TT\}$.
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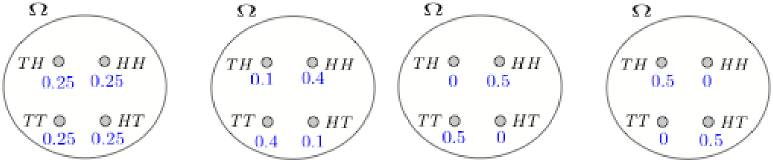

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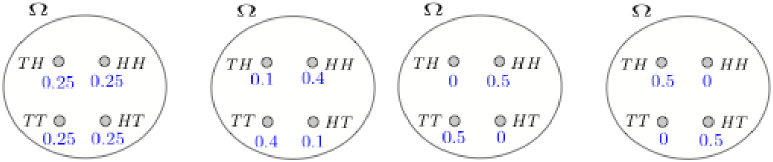

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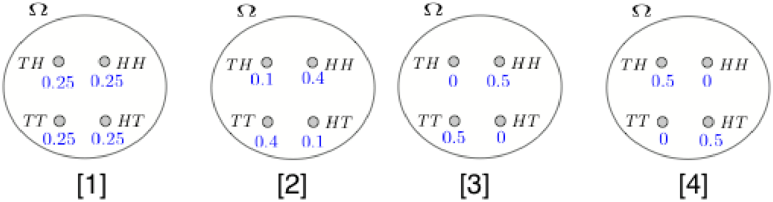

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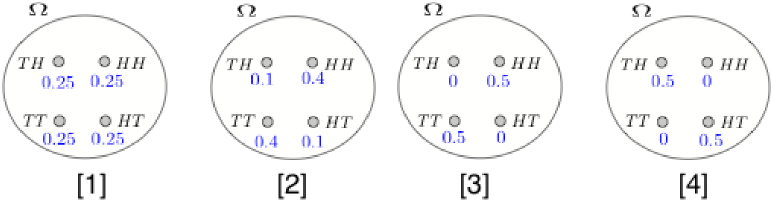

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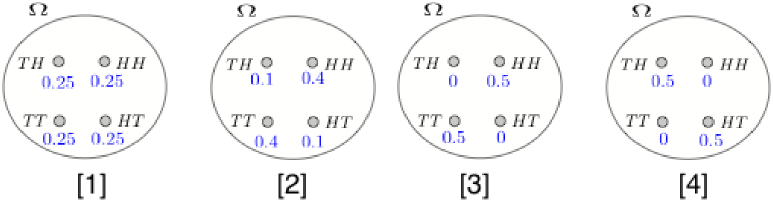

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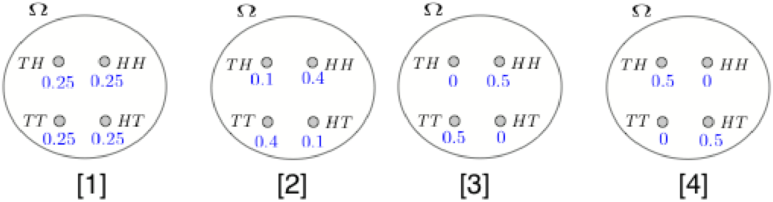

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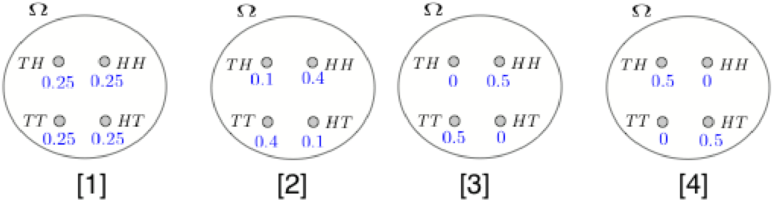

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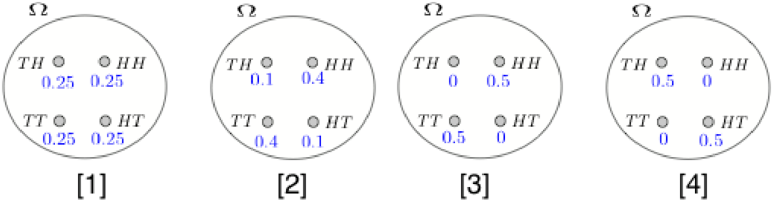

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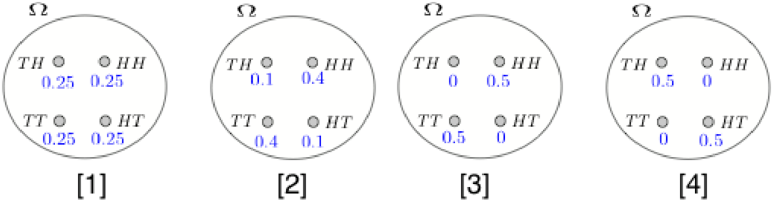

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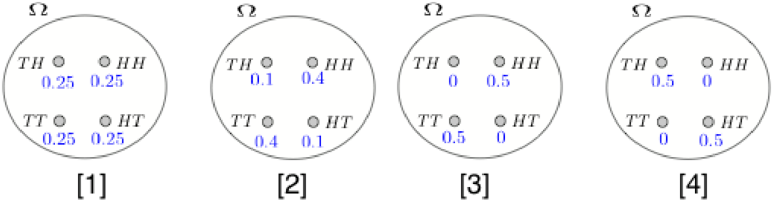

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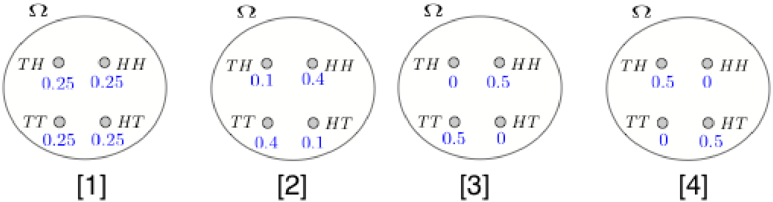

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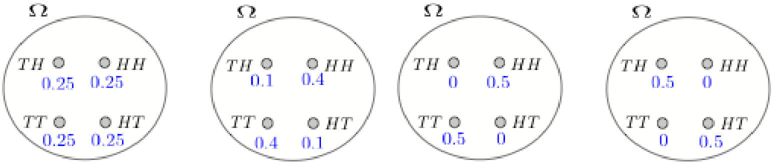

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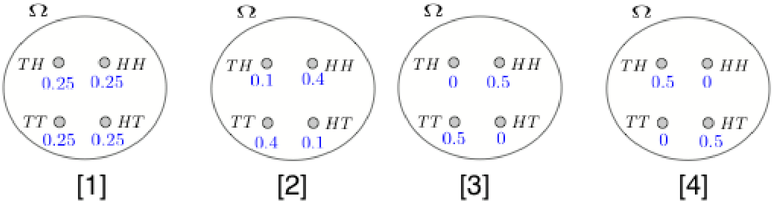

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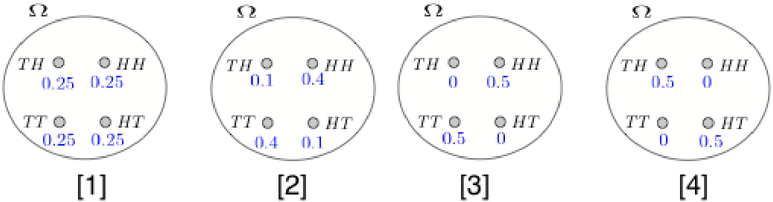

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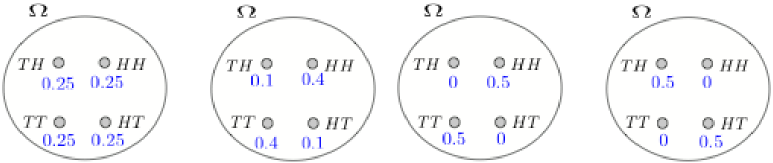

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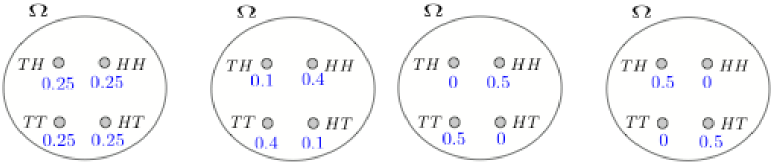

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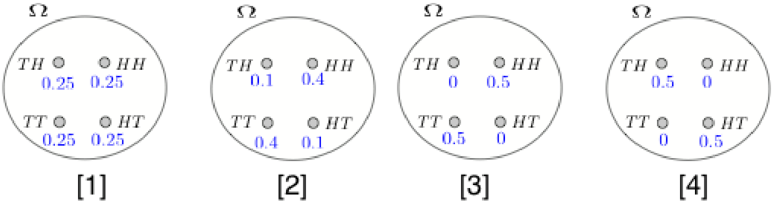

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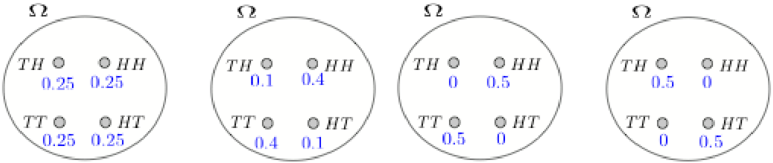

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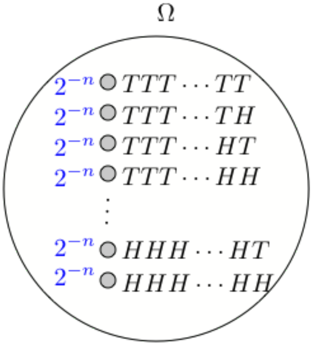

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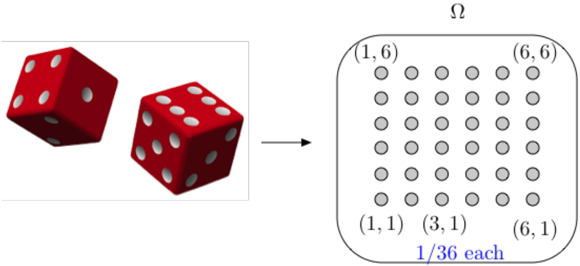

Physical Experiment   Probability Model

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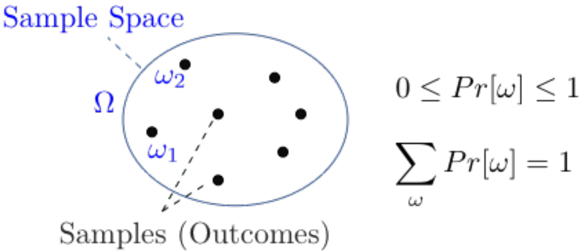

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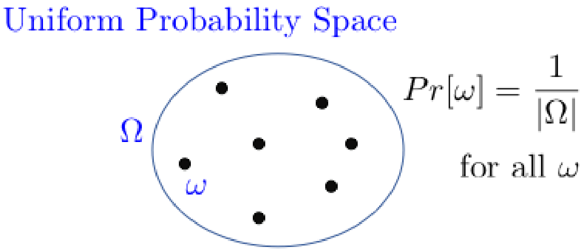

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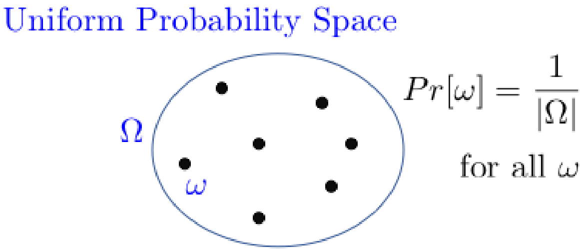

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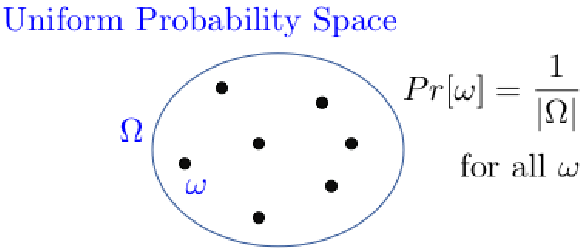

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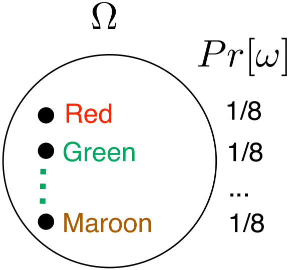

Probability model

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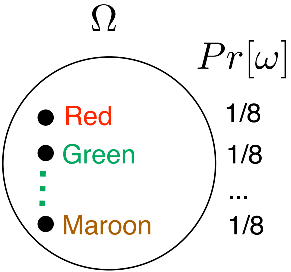

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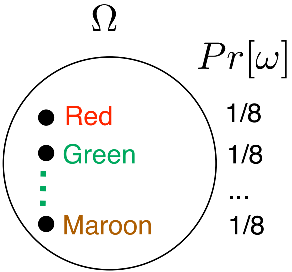

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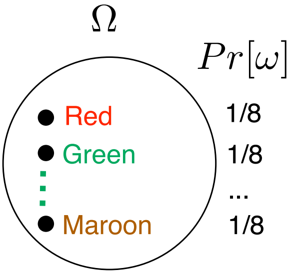

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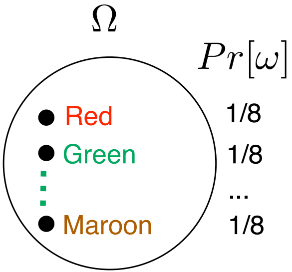

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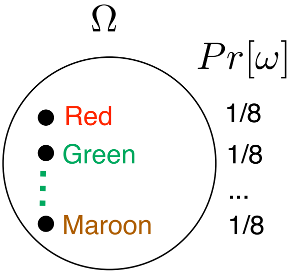

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Simplest physical model of a non-uniform probability space:

Probability Space: Formalism

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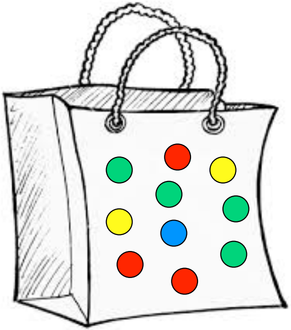
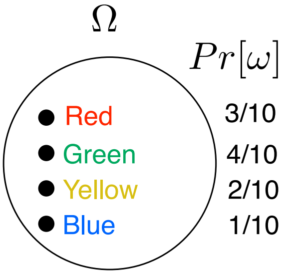


Physical experiment



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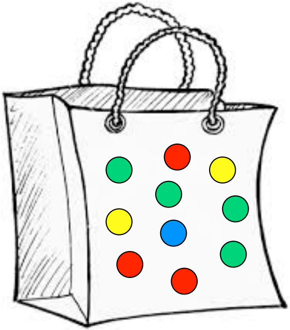

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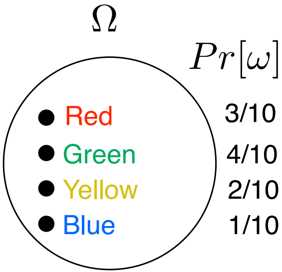

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$\Omega = \{$Red, Green, Yellow, Blue$\}$

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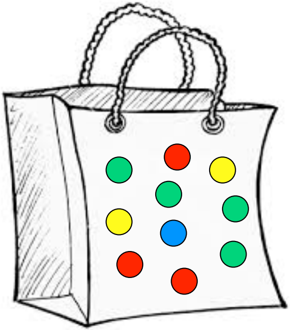

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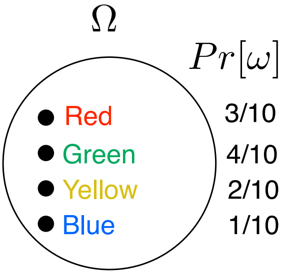

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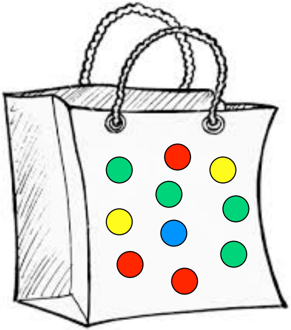

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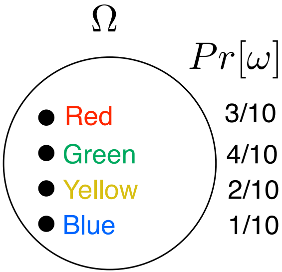

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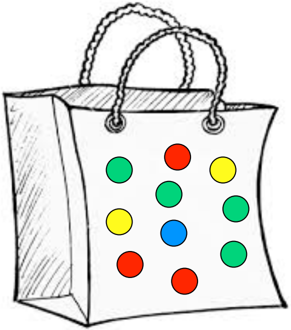

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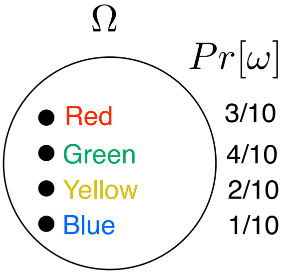

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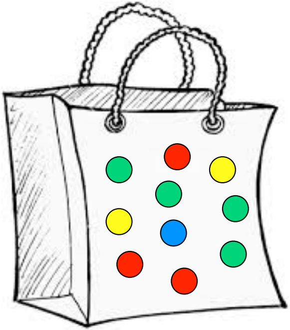

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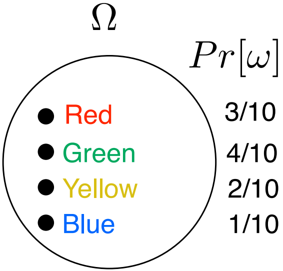

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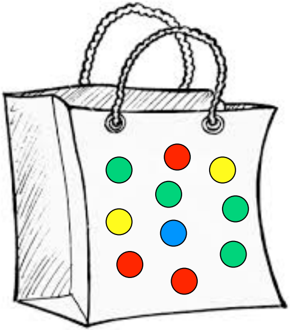

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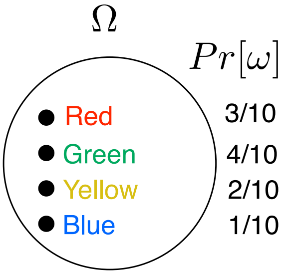

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Note: Probabilities are restricted to rational numbers: $\frac{N_k}{N}$.

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Physical model of a general non-uniform probability space:

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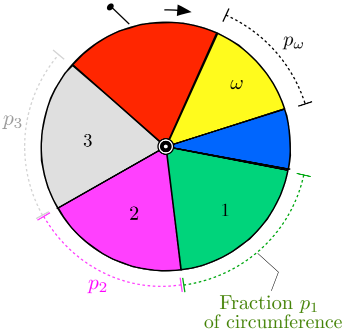

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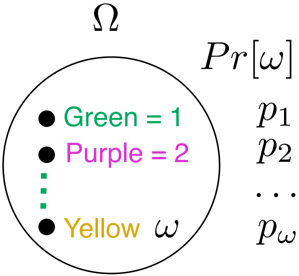

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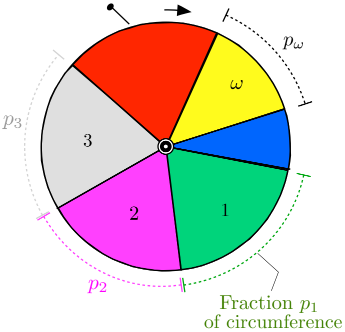

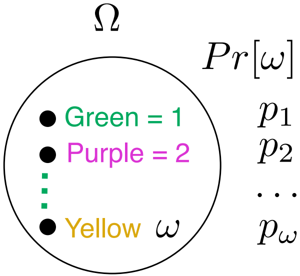

Physical experiment   Probability model

The roulette wheel stops in sector ω with probability p_ω.

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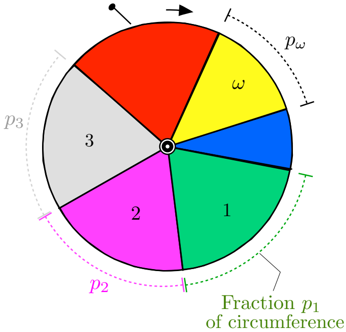

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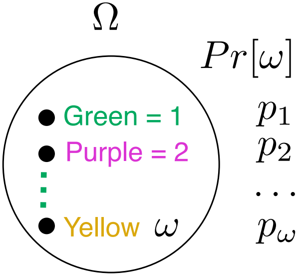

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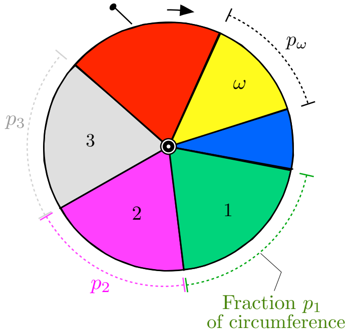

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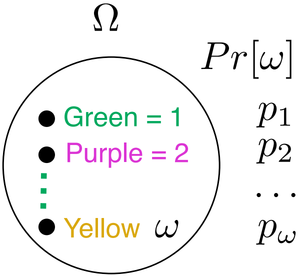

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Events

Next idea: an event!

Set notation review

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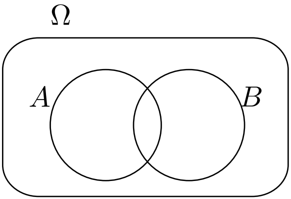
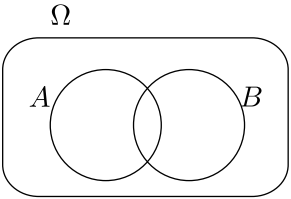


Figure : Two events

Set notation review



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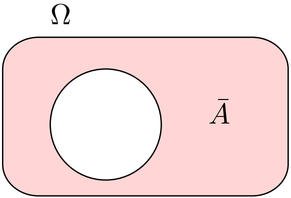

Figure : Complement (not)

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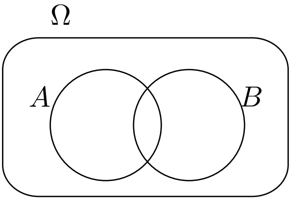

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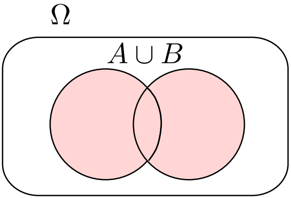
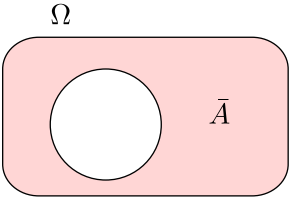


Figure : Union (or)



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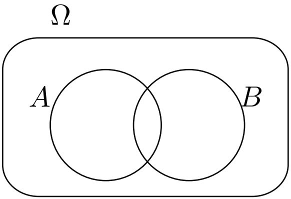

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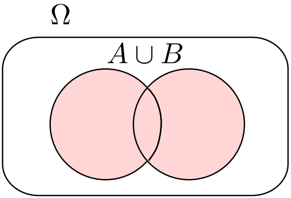

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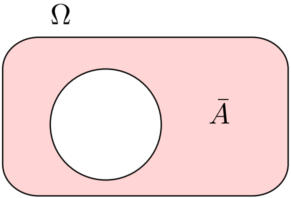

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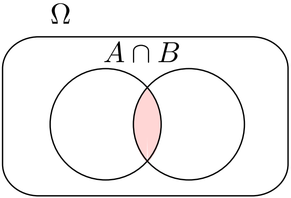
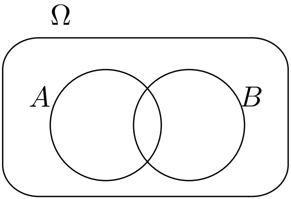


Figure : Intersection (and)

Set notation review



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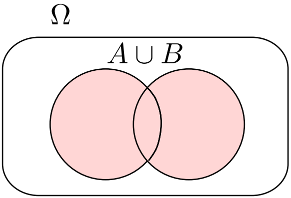

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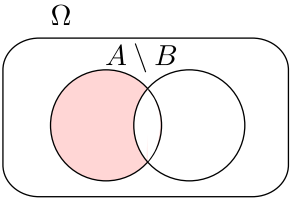
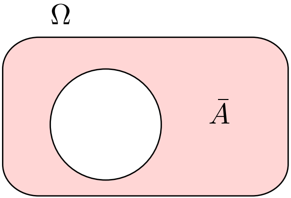


Figure : Difference (*A*, not *B*)



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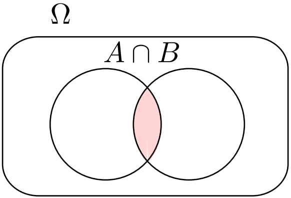

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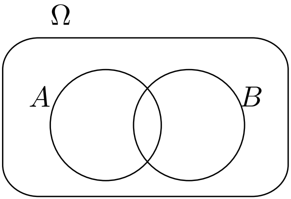

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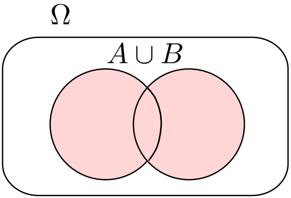

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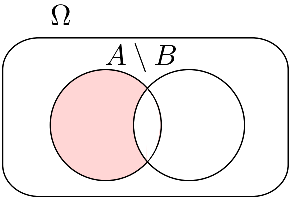

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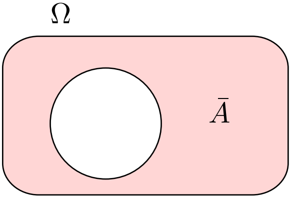

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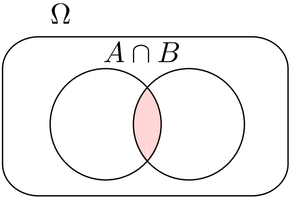

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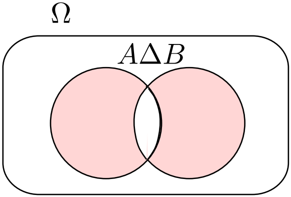

Figure : Symmetric difference (only one)

Probability of exactly one ‘heads’ in two coin flips?

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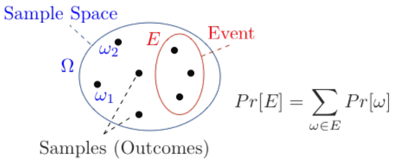

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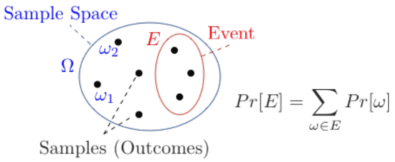

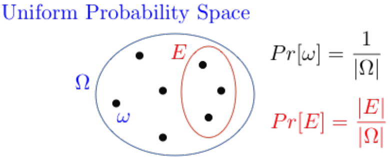

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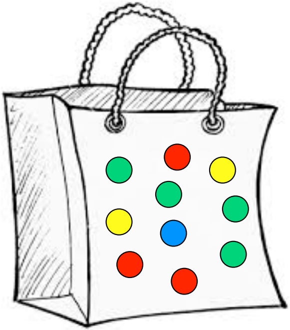

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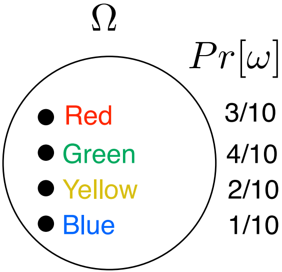

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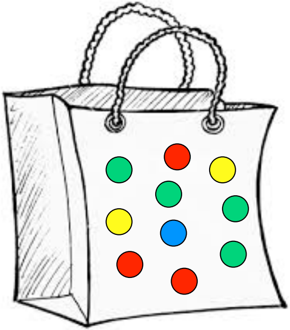

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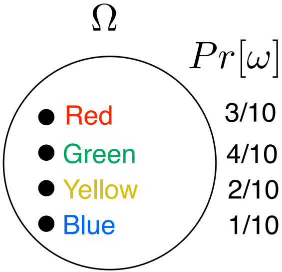

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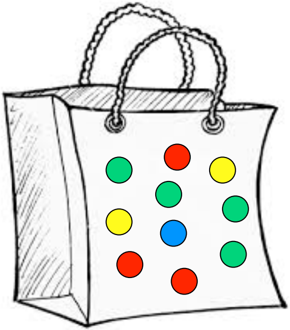

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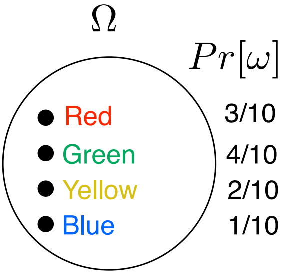

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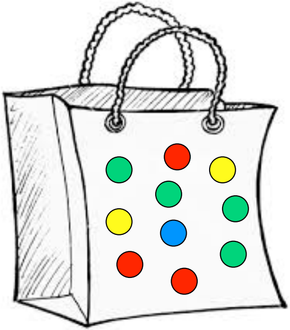

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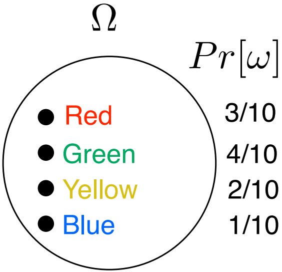
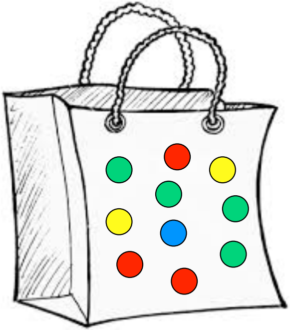


Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}] = \frac{3}{10},$$

Event: Example



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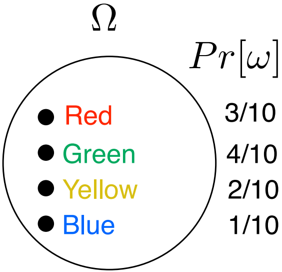

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$$Pr[\text{Red}] = \frac{3}{10}, Pr[\text{Green}] =$$

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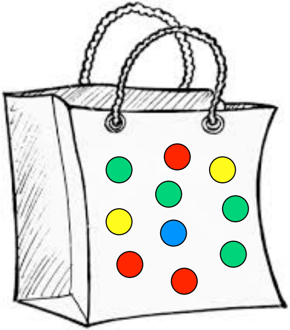

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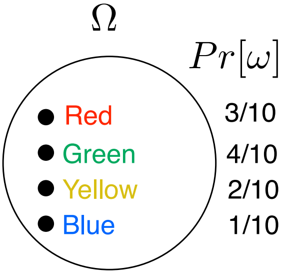

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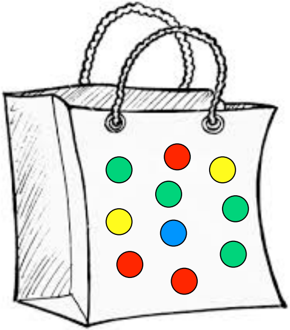

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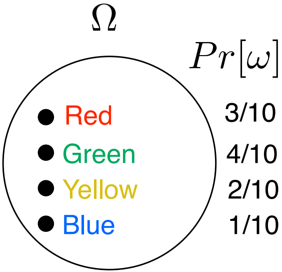

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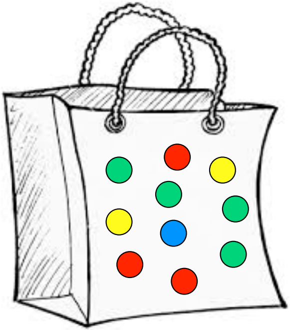

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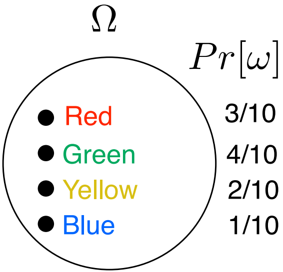

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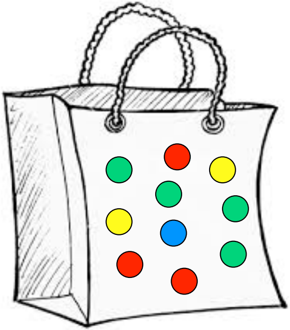

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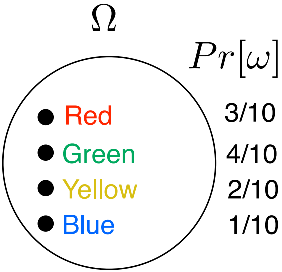

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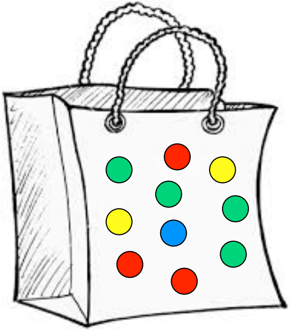

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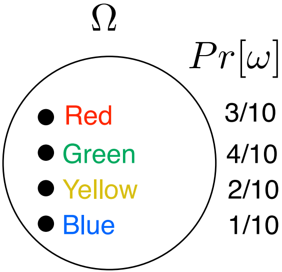

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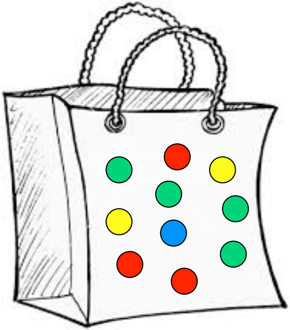

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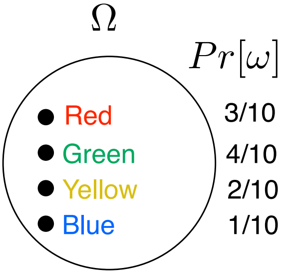

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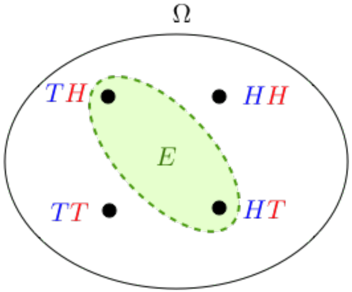

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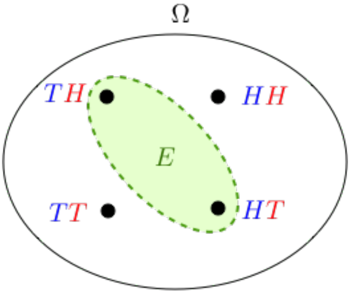

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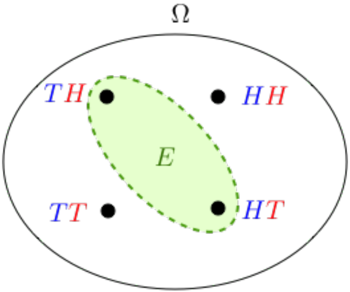

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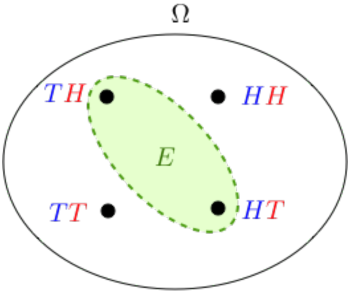

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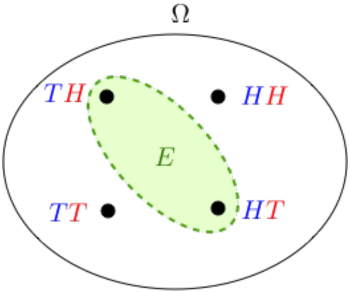

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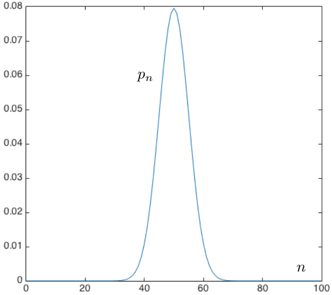

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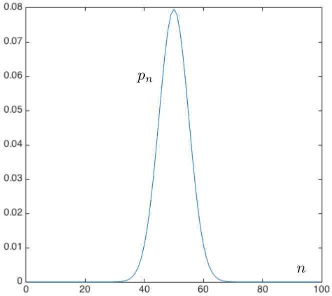

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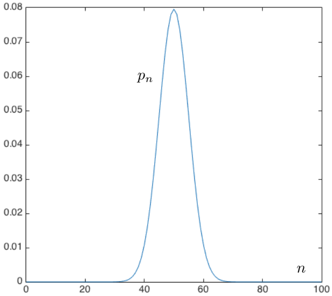

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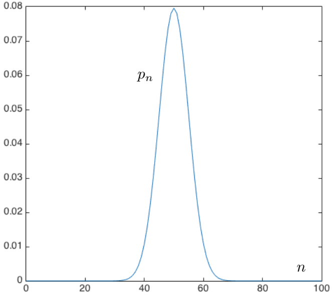

Event E_n = 'n heads'; $|E_n| = \binom{100}{n}$

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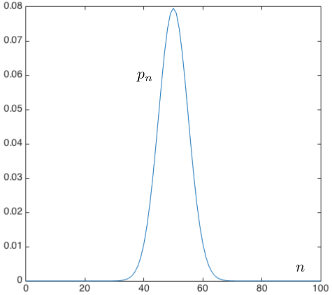

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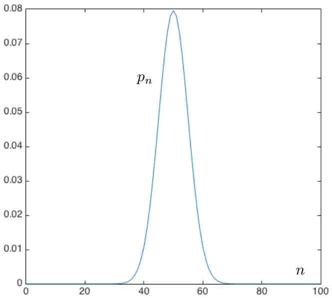

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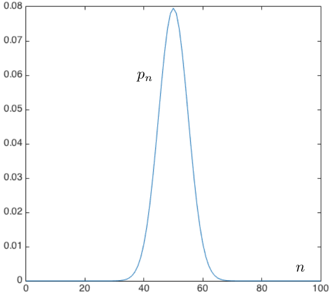

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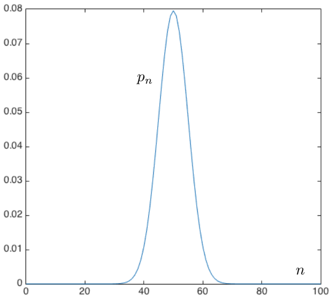

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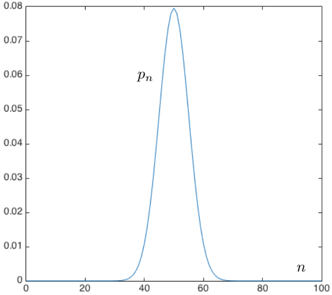

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Observe:

- Concentration around mean:

Probability of n heads in 100 coin tosses.

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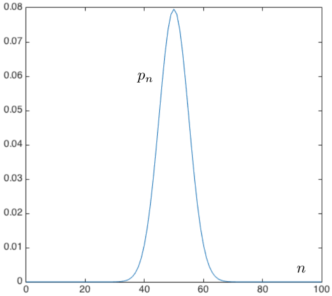

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Observe:

- Concentration around mean: Law of Large Numbers;

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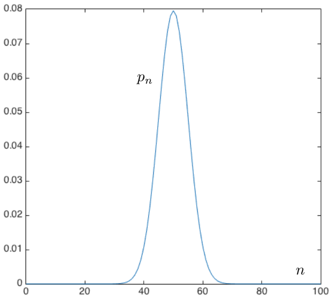

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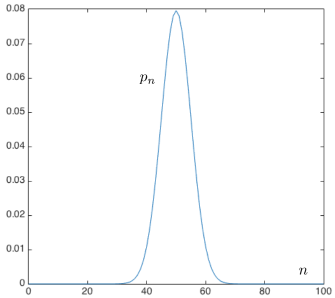

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Observe:

- Concentration around mean: Law of Large Numbers;
- Bell-shape: Central Limit Theorem.

Roll a red and a blue die.

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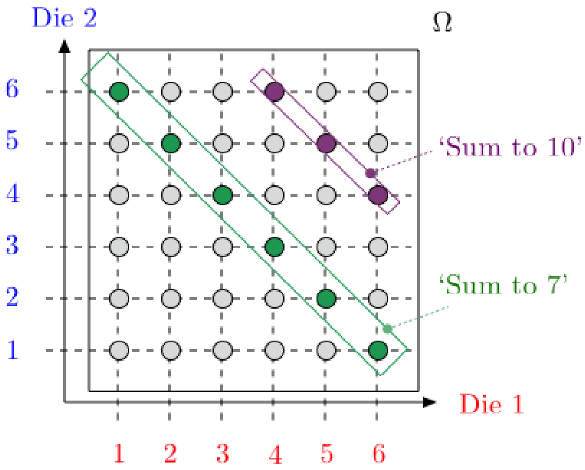

$$Pr[\text{Sum to } 7] = \frac{6}{36} \qquad Pr[\text{Sum to } 10] = \frac{3}{36}$$

Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses

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Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses $= \{H, T\}^{100}$.
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Uniform probability space: $Pr[\omega] = \frac{1}{2^{100}}$.

Event $E =$ "100 coin tosses with exactly 50 heads"

$|E|$?
Choose 50 positions out of 100 to be heads.

Exactly 50 heads in 100 coin tosses.

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Event $E =$ "100 coin tosses with exactly 50 heads"

$|E|$?
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$|E| = \binom{100}{50}$.

Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses $= \{H,T\}^{100}$.
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Event $E =$ "100 coin tosses with exactly 50 heads"

$|E|$?
Choose 50 positions out of 100 to be heads.
$|E| = \binom{100}{50}$.

$$Pr[E] = \frac{\binom{100}{50}}{2^{100}}.$$

Calculation.

Stirling formula (for large n):

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n .$$

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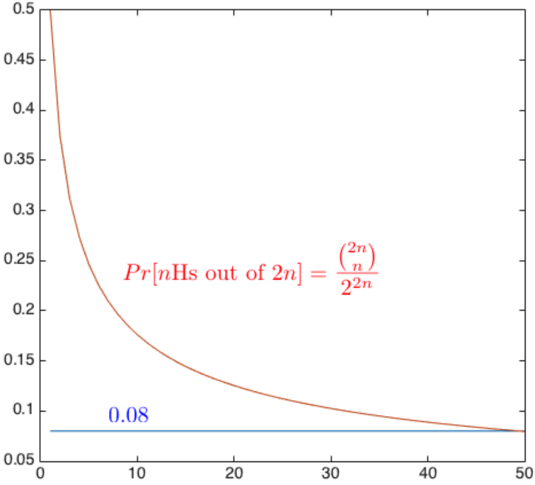
$$Pr[E] = \frac{|E|}{|\Omega|} = \frac{|E|}{2^{2n}} = \frac{1}{\sqrt{\pi n}} = \frac{1}{\sqrt{50\pi}} \approx .08.$$

Exactly 50 heads in 100 coin tosses.



Lecture 13: Summary

1. Random Experiment
2. Probability Space: $\Omega; Pr[\omega] \in [0,1]; \sum_{\omega} Pr[\omega] = 1$.
3. Uniform Probability Space: $Pr[\omega] = 1/|\Omega|$ for all $\omega \in \Omega$.
4. Events: subsets of Ω.
   $Pr[E] = \sum_{\omega \in E} Pr[\omega]$.