Today

Review for Midterm.

First there was logic...

A statement is a true or false.
Don't worry about Gödel.
Statements?
$3 = 4-1$? Statement!
$3 = 5$? Statement!
3 ? Not a statement!
$n = 3$? Not a statement...but a predicate.
Predicate: Statement with free variable(s).
Example: $x = 3$ Given a value for x, becomes a statement.
Predicate?
$n > 3$? Predicate: $P(n)$!
$x = y$? Predicate: $P(x,y)$!
$x+y$? No. An expression, not a statement.
Quantifiers:
$(\forall x)\ P(x)$. For every x, $P(x)$ is true.
$(\exists x)\ P(x)$. There exists an x, where $P(x)$ is true.

$(\forall n \in N), n^2 \geq n$: Any free variables? No. So it's a statement.
$(\forall x \in R)(\exists y \in R) y > x$.

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$, $A \implies B$.

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

Proofs: truth table or manipulation of known formulas.

$(\forall x \in \mathbb{R})(P(x) \wedge Q(x)) \equiv (\forall x \in \mathbb{R})P(x) \wedge (\forall x \in \mathbb{R})Q(x)$

If you think it's true:
Step 1: Show that when the thing on the left is true, the thing on the right is true. No matter what P and Q are!
Step 2: Show that when the thing on the right is true, the thing on the left is true. No matter what P and Q are!
Or manipulate the formulas.
If you think it's not true:
Find an example of $P(x)$ and $Q(x)$ such that one of the above steps fails.

...and then proofs...

Direct: $P \implies Q$
Example: a is even $\implies a^2$ is even.
Approach: What is even? $a = 2k$
$a^2 = 4k^2 = 2(2k^2)$
Integers closed under multiplication! So $2k^2$ is even.
a^2 is even.

Contrapositive: $P \implies Q$ or $\neg Q \implies \neg P$.
Example: a^2 is odd $\implies a$ is odd.
Contrapositive: a is even $\implies a^2$ is even.

Contradiction: P
$\neg P \implies$ **false**
Useful to prove something does not exist:
Example: rational representation of $\sqrt{2}$ does not exist.
Example: finite set of primes does not exist.
Example: rogue couple does not exist.

...jumping forward..

Contradiction in induction:
Find a place where induction step doesn't hold.
Something something Well ordering principle...

Contradiction in Stable Marriage:
First day where no woman improves. Does not exist.

Contradiction in Countability:
Assume there is a list with all the real numbers. Impossible.

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1)) \equiv (\forall n \in N)\ P(n)$.

Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

Induction on n.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n} - 1 = 8d)$

Induction Step: Prove $P(n+1)$

$$\begin{aligned} 3^{2n+2} - 1 &= 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\ &= 9(8d+1) - 1 \\ &= 72d + 8 \\ &= 8(9d+1) \end{aligned}$$

Divisible by 8.

$\square$

...Graphs...

$G = (V, E)$
V - set of vertices.
$E \subseteq V \times V$ - set of edges.

Directed: ordered pair of vertices.

Adjacent, Incident, Degree.
In-degree, Out-degree.

Thm: Sum of degrees is $2|E|$.
Edge is incident to 2 vertices.
Degree of vertices is total incidences.

Pair of Vertices are Connected:
If there is a path between them.

Connected Component: maximal set of connected vertices.

Connected Graph: one connected component.

Graph Algorithm: Eulerian Tour

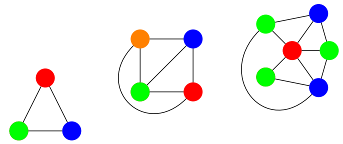
Thm: Every connected graph where every vertex has even degree has an Eulerian Tour; a tour which visits every edge exactly once.

Algorithm:
Take a walk using each edge at most once.
Property: return to starting point.
Why? Even degree. Always have an option

Remove the walk from the graph
Recurse on connected components.
Put together.
Property: walk visits every component.
Proof Idea: Original graph connected.

Graph Coloring.

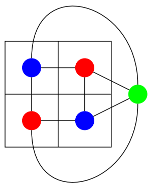
Given $G = (V, E)$, a coloring of a G assigns colors to vertices V where for each edge the endpoints have different colors.

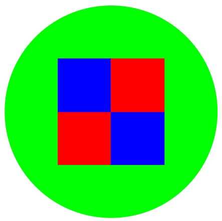


Notice that the last one, has one three colors.
Fewer colors than number of vertices.
Fewer colors than max degree node.

Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.





Four color theorem is about planar graphs!

Six color theorem.

Theorem: Every planar graph can be colored with six colors.

Proof:
Recall: $e \leq 3v - 6$ for any planar graph.
From Euler's Formula: $v + f = e + 2$.
$3f \leq 2e$

Total degree: $2e$
Average degree: $\leq \frac{2e}{v} \leq \frac{2(3v-6)}{v} \leq 6 - \frac{12}{v}$.

There exists a vertex with degree < 6 or at most 5.

Remove vertex v of degree at most 5.
Inductively color remaining graph.
Color is available for v since only five neighbors...
and only five colors are used. $\square$

Five color theorem

Theorem: Every planar graph can be colored with five colors.

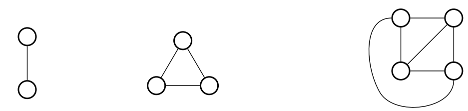
Proof: Not Today!

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Proof: Not Today!

Graph Types: Complete Graph.

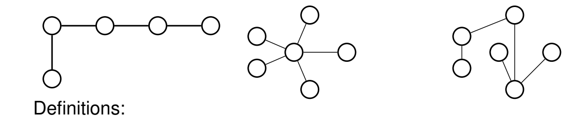


K_n, $|V| = n$

every edge present.
degree of vertex? $|V| - 1$.

Very connected.
Lots of edges: $n(n-1)/2$.
Wow.

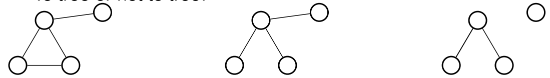
Trees.



Definitions:

A connected graph without a cycle.
A connected graph with $|V| - 1$ edges.
A connected graph where any edge removal disconnects it.
An acyclic graph where any edge addition creates a cycle.

To tree or not to tree!



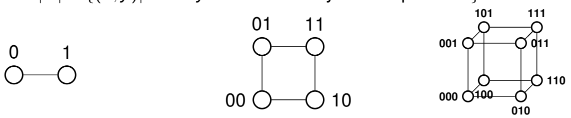
Minimally connected, minimum number of edges to connect.

Property:
Can remove a single node and break into components of size at most $|V|/2$.

Hypercube

Hypercubes. Really connected. $O(|V| \log |V|)$ edges!
Wait what? I thought it was $n2^{n-1}$. Oh... $2^n = |V|$...
Also represents bit-strings nicely.

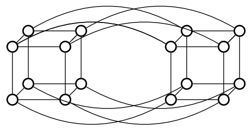
$G = (V, E)$
$|V| = \{0,1\}^n$,
$|E| = \{(x,y) | x$ and y differ in exactly one bit position.$\}$



Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.

An n-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$.



Hypercube:properties

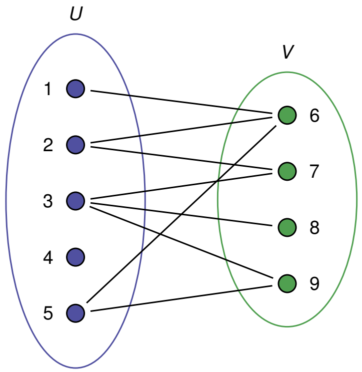
Dense cuts: Cutting off k nodes needs $\geq k$ edges.

FYI: Also cuts represent boolean functions. One side of the cut takes value 0. The other side takes value 1.

Nice Paths between nodes.
Get from 000100 to 101000.
000100 $\rightarrow$ 100100 $\rightarrow$ 101100 $\rightarrow$ 101000
Correct bits in string, moves along path in hypercube!

Good communication network!

Bipartite graphs



There is a cut with all the edges.

Cycles have length 4 or more edges.

Stable Marriage: a study in definitions and WOP.

n-men, n-women.

Each person has completely ordered preference list
contains every person of opposite gender.

Pairing/Marching.
Set of pairs (m_i, w_j) containing all people *exactly* once.
How many pairs? n.
People in pair are **partners** in pairing.

Rogue Couple in a pairing.
A m_j and w_k who like each other more than their partners

Stable Pairing.
Pairing with no rogue couples.

Does stable pairing exist?

Yes for matching.
No, for roommates problem.

TMA.

Traditional Marriage Algorithm:

Each Day:
Every man proposes to his favorite woman from the ones that haven't already rejected him.
Every woman rejects all but best man who proposes.

Useful Algorithmic Definitions:
Man **crosses off** woman who rejected him.
Woman's current proposer is **"on string."**

Key Property: Improvement Lemma:
Every day, if man on string for woman,
$\implies$ any future man on string is better. (proof by contradiction)

Stability: No rogue couple.
rogue couple (M,W)
$\implies$ M proposed to W
$\implies$ W ended up with someone she liked better than M.
Not rogue couple!

Optimality/Pessimal

Optimal partner if best partner in any stable pairing.
Not necessarily first in list.
Possibly no stable pairing with that partner.

Man-optimal pairing is pairing where every man gets optimal partner.

Thm: TMA produces male optimal pairing, S.
Proof by contradiction:
Let M be the first man to propose to someone worse than optimal partner W.
TMA: M asked W. And then got replaced by M'!
W prefers M'.
How much doesn M' like W?
Better than his match in optimal pairing? Impossible.
Worse than his match in the optimal pairing?
Then M wasn't the first!!

Thm: woman pessimal.

Man optimal $\implies$ Woman pessimal.
Woman optimal $\implies$ Man pessimal.

And then countability

More than one infinities

Some things are countable , like the natural numbers , or the rationals...
Why? There is a list!!
Some things are not countable , like the reals , or the set of all subsets of the naturals...
Why? **Diagonalization:** Well, assume there is a list. Can construct a diagonal element x. x is not in the list! Contradiction.

HALTING

The HALT problem: Is there a program that can tell you if another (generic) program halts on an input?

NO!

Why? Self reference!

Who cares? Using the same trick I can show that a bunch of problems are undecidable!
Like: Will this program P even print "Hello World"?
Or "Is there an input for this program P that will give an attacker admin access?

Counting!

Sample k items out of n.

	With Replacement	Without Replacement
Order matters	n^k	$\frac{n!}{(n-k)!}$
Order doesn't matter	$\binom{n+k-1}{n-1}$	$\binom{n}{k}$

Stars and bars!

Confusion yesterday: 10 hats. 7 days. I can wear the same hat on different days (replacement). I don't care which day I wore what (order doesn't matter).

Why is this stars and bars?

How many stars? One for each day. So 7

How many bars? One fewer than the hats. So 9

$||\star|\star\star|\star\star|||\star\star\star||$

Didn't wear hats 1 and 2. Wore hat 3 for 1 day, hat 4 for 2 days, hat 5 days. Didn't wear hats 6 and 7. Hat 8 for 3 days. Didn't wear hats 9 and 10.

Combinatorial Proofs.

Easy ones: $\binom{n}{k} = \binom{n}{n-k}$

Harder ones: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

What's the thing on the left? Number of subsets of size k of $\{1,2,\ldots,n+1\}$.

What's the thing on the right? Each subset either has, or doesn't have 1.

How many subsets of size k **have** 1? $k-1$ elements left to pick , from $\{2,\ldots,n+1\}$. $\binom{n}{k-1}$

How many subsets of size k **don't have** 1? k elements left to pick , from $\{2,\ldots,n+1\}$. $\binom{n}{k}$

Add them up. (**Sum rule**)

Midterm format

Time: 110 minutes.

Some short answers.
Get at ideas that you learned.
If something is taking too long maybe there is a trick!
Know material well: fast, correct.
Know material medium: slower, less correct.
Know material not so well: Uh oh.

Some longer questions.
Proofs, properties.
Not so much calculation.

Remember that a problem from hw and/or discussions is in the midterm! (identical or almost identical)

So study those!

FAQ

- Will this proof from the notes that I don't like be in the midterm?

  No.

- The why should I study it?

  Understanding a complex proof is a useful skill.

  Also, big proofs are usually a bunch of little proofs put together. And every proof is a new trick. And we like tricks!

Wrapup.

If you sent us an email about Midterm conflicts
Other arrangements.
Should have received an email from us.
You should know what to do by know.

Other issues....
email us.
Private message on piazza.

Good (sort of last minute) Studying!!!!!!!!!!!!!!!!