CS70: Counting

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July 7, 2016

Today:

- Balls and bins.
- Sum rule.
- Combinatorial proofs.
- Maybe start review?

What we've learned so far

Sample k items out of n.

	With Replacement	Without Replacement
Order matters	n^k	$\frac{n!}{(n-k)!}$
Order doesn't matter	$\binom{n+k-1}{n-1}$	$\binom{n}{k}$

A unifying example

Hats! Say I have 10 different hats. I'm thinking of how many different outfits I have for one week:

- Case 1: After I wear a hat I can wear it again. (Sampling with replacement)
  - Subcase: I care about which day I wore what: blue hat on Monday is different than blue hat on Tuesday. duh. (Order matters.)
    10 options for Monday, 10 options for Tuesday... 10^7.
  - Subcase: I don't care about which day I wore what, I just care which hats I ending up wearing. (Order doesn't matter.)
    How many samples? a day is a sample, so 7. From how big of a set? 10 hats.
    $\binom{n+k-1}{n-1} = \binom{10+7-1}{10-1} = \binom{16}{9} = 11440$

A unifying example

- Case 2: After I wear a hat I destroy it. (Sampling without replacement)
  - Subcase: I care about which day I wore what: blue hat on Monday is different than blue hat on Tuesday. (Order matters.)
    10 options for Monday, 9 options for Tuesday...
    $\frac{n!}{(n-k)!} = \frac{10!}{3!} = 604800$.
  - Subcase: I don't care about which day I wore what. (Order doesn't matter.)
    $\binom{10}{7} = 120$

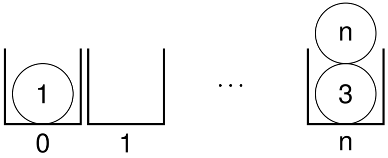
A cool stars and bars application

How many (non-negative) solutions to $x + y = 10$?
Easy: $x = 0, y = 10$, $x = 1, y = 9$,$x = 2, y = 8$, $\ldots$, $x = 10, y = 0$. So 11 solutions.
Same as 10 stars, and 1 bar.

$x = 3, y = 7: \quad \star\star\star|\star\star\star\star\star\star\star$
Think of a star as the number 1.

How many ways to make an 8 problem midterm such the total points add up to 100?
100 stars, 7 bars.

Balls in bins.



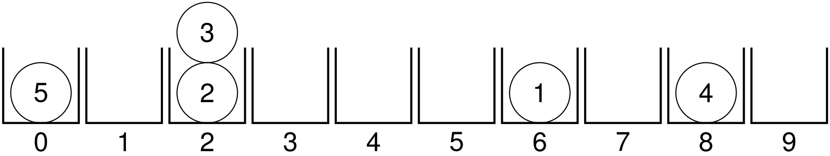
"k Balls in n bins" $\equiv$ "k samples from n possibilities."

"indistinguishable balls" $\equiv$ "order doesn't matter"

"only one ball in each bin" $\equiv$ "without replacement"

Balls and bins

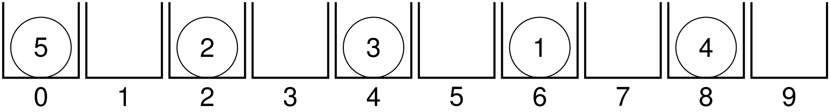
How many 5 digit numbers?
Throwing 5 numbered balls in 10 (numbered) bins:



Picture has number 62280.
5 samples from 10 possibilities with replacement (order matters): 10^5

Balls and bins

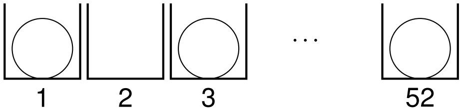
How many 5 digit numbers without repeating a digit?
Throwing 5 numbered balls in 10 (numbered) bins, at most one ball per bin:



Picture has number 62480.
5 samples from 10 possibilities without replacement (order matters):
$\frac{10!}{5!}$

Balls and bins

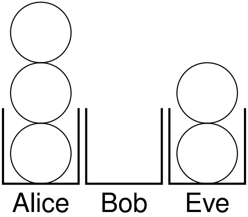
How many 3 card poker hands?
Throwing 3 indistinguishable balls in 52 (numbered) bins. At most 1 ball per bin:



Picture has cards 1,3 and 52.
3 samples from 52 possibilities without replacement (order doesn't matter): $\binom{52}{3}$

Balls and bins

Dividing 5 dollars among Alice, Bob and Eve.
5 indistinguishable balls into 3 (numbered) bins:



Picture: Alice 3, Bob 0, Eve 2.
5 samples from 3 possibilities with replacement (order doesn't matter): $\binom{7}{2}$

Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?
Sum rule: Can sum over disjoint sets.
No jokers or exactly one joker or exactly two jokers

$$\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.$$

Two distinguishable jokers in 54 card deck.
How many 5 card poker hands (distinguishable jokers)?
No jokers or exactly one of two jokers or exactly two jokers

$$\binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$$

Wait a minute! Same as choosing 5 cards from 54 or

$$\binom{54}{5}$$

Theorem: $\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.
Proof: Above is combinatorial proof.

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}$, $\binom{52}{5} = \frac{52!}{5!47!}$, $\binom{52}{4} = \frac{52!}{4!48!}$, $\binom{52}{3} = \frac{52!}{3!49!}$

$RHS = \frac{52!}{5!47!} + 2\frac{52!}{4!48!} + \frac{52!}{3!49!} = \frac{52!(4!48!3!49!+2*5!47!3!49!+5!47!4!48!)}{5!47!4!48!3!49!} \stackrel{?}{=} \frac{54!}{5!49!}$

49! and 5! cancel out. Cross multiply and get:

$54!47!4!48!3! \stackrel{?}{=} 52!(4!48!3!49! + 2 * 5!47!3!49! + 5!47!4!48!)$

I tried this for a while.......

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same
as choosing $n-k$ elements to not take.
$\implies \binom{n}{n-k}$ subsets of size k.

□

Pascal's Triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1
Second row: $(1+x)^2 = 1+2x+x^2$. Coefficients: 1,2 and 1
Third row: $(1+x)^3 = 1+3x+3x^2+x^3$. Coefficients: 1,3,3 and 1
.....
Foil?? I hate this already...

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.
Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.
$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth. 6 of them , aka $\binom{4}{2}$.

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
$$

Pascal's rule $\Longrightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Pick the first.Then I need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$
How many don't contain the first element ?
Need to choose k elements from remaining n elements.
$\implies \binom{n}{k}$
So, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. □

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the smallest element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is smallest element chosen: $\binom{n-1}{k-1}$ choices for the rest.
2 is smallest element chosen: $\binom{n-2}{k-1}$ choices for the rest.
and so on.

Add them up to get the total number of subsets of size k
which is also $\binom{n}{k}$.

$\square$

Binomial Theorem: $x = 1$

Theorem: $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$

Proof: How many subsets of $\{1, \ldots, n\}$?
Construct a subset with sequence of n choices:
  element i **is in** or **is not** in the subset: 2 possibilities.
First rule of counting: $2 \times 2 \cdots \times 2 = 2^n$ subsets.

How many subsets of $\{1, \ldots, n\}$?
  $\binom{n}{i}$ = subsets of size i.
  A subset has size either 0, or 1, or 2, ..., or n
Sum over i to get total number of subsets. $\square$

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
Used to reason about all subsets
by adding number of subsets of size 0, 1, 2, 3,...

Also reasoned about subsets that contained
or didn't contain an element. (E.g., first element, first i elements.)

Inclusion/Exclusion Rule: For any S and T,
$|S \cup T| = |S| + |T| - |S \cap T|$.

Example: How many 10-digit phone numbers have 7 as their first or second digit?

S = phone numbers with 7 as first digit. $|S| = 10^9$

T = phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T$ = phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

Summary.

Inclusion/Exclusion: two sets of objects.
Add number of each subtract intersection of sets.
Sum Rule: If disjoint just add.

Combinatorial Proofs: Identity from counting same in two ways.
Pascal's Triangle Example: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.
RHS: Number of subsets of $n+1$ items size k.
LHS: $\binom{n}{k-1}$ counts subsets of $n+1$ items with first item.
$\binom{n}{k}$ counts subsets of $n+1$ items without first item.
Disjoint – so add!