# CS70: Counting

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Reminder: Don't write on the board.

What's to come?

What's to come? Probability.

What's to come? Probability. A bag contains:

What's to come? Probability.

A bag contains:



What's to come? Probability. A bag contains:

What is the chance that a ball taken from the bag is blue?  $\frac{3}{8}$ .

What's to come? Probability. A bag contains:

What is the chance that a ball taken from the bag is blue?  $\frac{3}{8}$ . How did I know?

What's to come? Probability.

A bag contains:



What is the chance that a ball taken from the bag is blue?  $\frac{3}{8}$ . How did I know?

Count blue.

What's to come? Probability. A bag contains:



What is the chance that a ball taken from the bag is blue?  $\frac{3}{8}$ . How did I know?

Count blue. Count total.

What's to come? Probability. A bag contains:



# What is the chance that a ball taken from the bag is blue? $\frac{3}{8}$ . How did I know?

Count blue. Count total. Divide.

What's to come? Probability.

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What is the chance that a ball taken from the bag is blue?  $\frac{3}{8}$ . How did I know?

Count blue. Count total. Divide.

Today (and tomorrow):

What's to come? Probability.

A bag contains:



What is the chance that a ball taken from the bag is blue?  $\frac{3}{8}$ . How did I know?

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Today (and tomorrow): Counting!

What's to come? Probability.

A bag contains:



What is the chance that a ball taken from the bag is blue?  $\frac{3}{8}$ . How did I know?

Count blue. Count total. Divide.

Today (and tomorrow): Counting!

Next week: Probability.

What's to come? Probability.

A bag contains:



What is the chance that a ball taken from the bag is blue?  $\frac{3}{8}$ . How did I know?

Count blue. Count total. Divide.

Today (and tomorrow): Counting!

Next week: Probability.

Make sure you understand counting if you want to understand probability!!!

# **Outline: basics**

- 1. Counting.
- 2. Rules of Counting.
- 3. Sample with/without replacement where order does/doesn't matter.
- 4. Combinatorial proofs (mostly tomorrow)

1 + 1 = ?



1 + 1 = ? 2

1+1=? 2 3+4=?

 $\begin{array}{ll} 1+1=? & 2 \\ 3+4=? & 7 \end{array}$ 

 $\begin{array}{ccc} 1+1=? & 2\\ 3+4=? & 7\\ How many 100\mbox{-bit strings are there that contain exactly 6 ones?}\\ 1,192,052,400 \end{array}$ 

 $\begin{array}{ccc} 1+1=? & 2\\ 3+4=? & 7\\ How many 100\mbox{-bit strings are there that contain exactly 6 ones?}\\ 1,192,052,400 \end{array}$ 

How many outcomes possible for *k* coin tosses? How many poker hands? How many handshakes for *n* people? How many 10 digit numbers? How many 10 digit numbers without repetition?

How many 3-bit strings?

How many 3-bit strings? (I know, I know...

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How many 3-bit strings? (I know, I know... Calm down....) How many different sequences of three bits from  $\{0,1\}$ ?

How many 3-bit strings? (I know, I know... Calm down....) How many different sequences of three bits from  $\{0,1\}$ ? How would you make one sequence?

How many 3-bit strings? (I know, I know... Calm down....) How many different sequences of three bits from  $\{0,1\}$ ? How would you make one sequence? Pick the first digit.












Objects made by choosing from  $n_1$  options

Objects made by choosing from  $n_1$  options, then  $n_2$  options

Objects made by choosing from  $n_1$  options, then  $n_2$  options, ..., then  $n_k$  options:

Objects made by choosing from  $n_1$  options, then  $n_2$  options, ..., then  $n_k$  options: the number of objects is  $n_1 \times n_2 \cdots \times n_k$ .

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*n*<sub>1</sub>

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In picture,  $2 \times 2 \times 3 = 12$ 

Objects made by choosing from  $n_1$  options, then  $n_2$  options , ..., then  $n_k$  options: the number of objects is  $n_1 \times n_2 \cdots \times n_k$ .



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How many outcomes possible for k coin tosses?

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2 ways for first choice,

How many outcomes possible for *k* coin tosses? 2 ways for first choice, 2 ways for second choice, ...

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How many outcomes possible for *k* coin tosses? 2 ways for first choice, 2 ways for second choice, ...  $2 \times 2$ 

How many outcomes possible for *k* coin tosses? 2 ways for first choice, 2 ways for second choice, ...  $2 \times 2 \cdots$ 

How many outcomes possible for *k* coin tosses? 2 ways for first choice, 2 ways for second choice, ...  $2 \times 2 \cdots \times 2$ 

How many outcomes possible for *k* coin tosses? 2 ways for first choice, 2 ways for second choice, ...  $2 \times 2 \cdots \times 2 = 2^k$ 

How many outcomes possible for k coin tosses?

2 ways for first choice, 2 ways for second choice, ...  $2 \times 2 \cdots \times 2 = 2^k$ 

How many k digit numbers (in decimal)?

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How many k digit numbers (in decimal)?

10 ways for first choice,

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How many *k* digit numbers (in decimal)?

10 ways for first choice, 10 ways for second choice, ...

How many outcomes possible for k coin tosses?

2 ways for first choice, 2 ways for second choice, ...  $2 \times 2 \cdots \times 2 = 2^k$ 

How many *k* digit numbers (in decimal)?

10 ways for first choice, 10 ways for second choice,  $\dots$  10

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How many *k* digit numbers (in decimal)?

10 ways for first choice, 10 ways for second choice, ... 10  $\times$ 

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How many *k* digit numbers (in decimal)?

10 ways for first choice, 10 ways for second choice, ...  $10\times10\cdots$ 

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2 ways for first choice, 2 ways for second choice, ...  $2 \times 2 \cdots \times 2 = 2^k$ 

How many *k* digit numbers (in decimal)?

10 ways for first choice, 10 ways for second choice, ...  $10 \times 10 \cdots \times 10$ 

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2 ways for first choice, 2 ways for second choice, ...  $2 \times 2 \cdots \times 2 = 2^k$ 

How many *k* digit numbers (in decimal)?

10 ways for first choice, 10 ways for second choice, ...  $10 \times 10 \cdots \times 10 = 10^k$ 

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How many k digit numbers (in decimal)?

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How many *n* digit base *m* numbers?

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How many k digit numbers (in decimal)?

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How many *n* digit base *m* numbers?

*m* ways for first,

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2 ways for first choice, 2 ways for second choice, ...  $2 \times 2 \cdots \times 2 = 2^k$ 

How many *k* digit numbers (in decimal)?

10 ways for first choice, 10 ways for second choice, ...  $10 \times 10 \cdots \times 10 = 10^k$ 

How many *n* digit base *m* numbers?

m ways for first, m ways for second, ...

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2 ways for first choice, 2 ways for second choice, ...  $2 \times 2 \cdots \times 2 = 2^k$ 

How many k digit numbers (in decimal)?

10 ways for first choice, 10 ways for second choice, ...  $10 \times 10 \cdots \times 10 = 10^k$ 

How many *n* digit base *m* numbers?

m ways for first, m ways for second, ...  $m^n$ 

# Functions, polynomials.

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....|*T*|<sup>|S|</sup>

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p ways to choose for first coefficient, p ways for second, ...  $...p^{d+1}$ 

How many 10 digit numbers?

<sup>1</sup>By definition: 0! = 1.

How many 10 digit numbers? 10<sup>10</sup>.

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How many 10 digit numbers without repeating a digit?

How many 10 digit numbers? 10<sup>10</sup>. How many 10 digit numbers **without repeating a digit**? 10 ways for first,

How many 10 digit numbers? 10<sup>10</sup>.

How many 10 digit numbers without repeating a digit?

10 ways for first, 9 ways for second,

How many 10 digit numbers? 10<sup>10</sup>.

How many 10 digit numbers without repeating a digit?

10 ways for first, 9 ways for second, 8 ways for third,

How many 10 digit numbers? 10<sup>10</sup>.

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How many 10 digit numbers? 10<sup>10</sup>. How many 10 digit numbers **without repeating a digit**? 10 ways for first, 9 ways for second, 8 ways for third, ...

...  $10 * 9 * 8 \cdots * 1 = 10!^{1}$ 

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How many orderings of *n* objects are there? **Permutations of** *n* **objects.** 

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How many orderings of *n* objects are there? **Permutations of** *n* **objects.** 

*n* ways for first, n-1 ways for second,

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How many orderings of *n* objects are there? **Permutations of** *n* **objects.** 

*n* ways for first, n-1 ways for second, n-2 ways for third, ...

... 
$$n * (n-1) * (n-2) \cdot *1 = n!$$

#### <sup>1</sup>By definition: 0! = 1.

How many one-to-one functions from S to S.

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How many one-to-one functions from S to S.

|S| choices for  $f(s_1)$ , |S| - 1 choices for  $f(s_2)$ , ... So total number is  $|S| \times (|S| - 1) \cdots 1 = |S|!$ A one-to-one function is a permutation!

How many poker hands? (5 cards)

<sup>&</sup>lt;sup>2</sup>When each unordered object corresponds to an equal numbers of ordered objects.

How many poker hands? (5 cards)

52

<sup>&</sup>lt;sup>2</sup>When each unordered object corresponds to an equal numbers of ordered objects.

How many poker hands? (5 cards)  $52 \times 51$ 

<sup>&</sup>lt;sup>2</sup>When each unordered object corresponds to an equal numbers of ordered objects.

How many poker hands? (5 cards)  $52 \times 51 \times 50$ 

<sup>&</sup>lt;sup>2</sup>When each unordered object corresponds to an equal numbers of ordered objects.

How many poker hands? (5 cards)  $52 \times 51 \times 50 \times 49$ 

<sup>&</sup>lt;sup>2</sup>When each unordered object corresponds to an equal numbers of ordered objects.

How many poker hands? (5 cards)  $52 \times 51 \times 50 \times 49 \times 48$ 

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How many poker hands? (5 cards)  $52 \times 51 \times 50 \times 49 \times 48$  ???

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Aren't A, K, Q, 10, J of spades

<sup>&</sup>lt;sup>2</sup>When each unordered object corresponds to an equal numbers of ordered objects.

How many poker hands? (5 cards)

 $52 \times 51 \times 50 \times 49 \times 48$  ???

Aren't A, K, Q, 10, J of spades and 10, J, Q, K, A of spades the same?

<sup>&</sup>lt;sup>2</sup>When each unordered object corresponds to an equal numbers of ordered objects.

How many poker hands? (5 cards)

 $52\times51\times50\times49\times48$  ???

Aren't A, K, Q, 10, J of spades and 10, J, Q, K, A of spades the same? **Second Rule of Counting:** If order doesn't matter count ordered objects and then divide by number of orderings.<sup>2</sup>

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Number of orderings for a poker hand: 5!

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 $\frac{52 \times 51 \times 50 \times 49 \times 48}{5!}$ 

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How many poker hands? (5 cards)

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Number of orderings for a poker hand: 5!

Can write as	$52 \times 51 \times 50 \times 49 \times 48$
	5!
	52!
	$\overline{5! \times 47!}$

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How many poker hands? (5 cards)

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Can write as	$52 \times 51 \times 50 \times 49 \times 48$
	5!
	52!
	$\overline{5! \times 47!}$

Generic: ways to choose 5 out of 52 possibilities.

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How many red nodes (ordered objects)?

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How many red nodes (ordered objects)? 9.

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How many red nodes (ordered objects)? 9.

How many red nodes mapped to one blue node?

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How many blue nodes (unordered objects)?

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How many red nodes (ordered objects)? 9.

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How many blue nodes (unordered objects)?  $\frac{9}{3}$ 

**Second Rule of Counting:** If order doesn't matter count ordered objects and then divide by number of orderings.



How many red nodes (ordered objects)? 9.

How many red nodes mapped to one blue node? 3.

How many blue nodes (unordered objects)?  $\frac{9}{3} = 3$ .

**Second Rule of Counting:** If order doesn't matter count ordered objects and then divide by number of orderings.



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How many blue nodes (unordered objects)?  $\frac{9}{3} = 3$ .

If you know:(1) how many red vertices

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How many blue nodes (unordered objects)?  $\frac{9}{3} = 3$ .

If you know:(1) how many red vertices and (2) in-degree for each blue vertex.

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Then, you know how many blue vertices!

**Second Rule of Counting:** If order doesn't matter count ordered objects and then divide by number of orderings.



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If you know:(1) how many red vertices and (2) in-degree for each blue vertex.

Then, you know how many blue vertices!

How many poker deals? (red vertices)

**Second Rule of Counting:** If order doesn't matter count ordered objects and then divide by number of orderings.



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How many blue nodes (unordered objects)?  $\frac{9}{3} = 3$ .

If you know:(1) how many red vertices and (2) in-degree for each blue vertex.

Then, you know how many blue vertices!

How many poker deals? (red vertices) 52.51.50.49.48.

**Second Rule of Counting:** If order doesn't matter count ordered objects and then divide by number of orderings.



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How many red nodes mapped to one blue node? 3.

How many blue nodes (unordered objects)?  $\frac{9}{3} = 3$ .

If you know:(1) how many red vertices and (2) in-degree for each blue vertex.

Then, you know how many blue vertices!

How many poker deals? (red vertices) 52 · 51 · 50 · 49 · 48.

How many poker hands per deal? (degree)

**Second Rule of Counting:** If order doesn't matter count ordered objects and then divide by number of orderings.



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How many red nodes mapped to one blue node? 3.

How many blue nodes (unordered objects)?  $\frac{9}{3} = 3$ .

If you know:(1) how many red vertices and (2) in-degree for each blue vertex.

Then, you know how many blue vertices!

How many poker deals? (red vertices)  $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ .

How many poker hands per deal? (degree) Map each deal to ordered deal.

**Second Rule of Counting:** If order doesn't matter count ordered objects and then divide by number of orderings.



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How many poker hands per deal? (degree) Map each deal to ordered deal. 5!

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How many poker hands?

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How many poker deals? (red vertices) 52 · 51 · 50 · 49 · 48.

How many poker hands per deal? (degree) Map each deal to ordered deal. 5!

How many poker hands?  $\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!}$ 

$$n \times (n-1)$$

$$\frac{n \times (n-1)}{2}$$

$$\frac{n\times(n-1)}{2} = \frac{n!}{(n-2)!\times 2}$$

Choose 2 out of n?

$$\frac{n\times(n-1)}{2}=\frac{n!}{(n-2)!\times 2}$$

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$$n \times (n-1) \times (n-2)$$

Choose 2 out of n?

$$\frac{n\times(n-1)}{2}=\frac{n!}{(n-2)!\times 2}$$

$$\frac{n\times(n-1)\times(n-2)}{3!}$$

Choose 2 out of n?

$$\frac{n\times(n-1)}{2}=\frac{n!}{(n-2)!\times 2}$$

$$\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}$$

Choose 2 out of n?

$$\frac{n\times(n-1)}{2}=\frac{n!}{(n-2)!\times 2}$$

Choose 3 out of n?

$$\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}$$

Choose k out of n?

*n*!

Choose 2 out of n?

$$\frac{n\times(n-1)}{2}=\frac{n!}{(n-2)!\times 2}$$

Choose 3 out of n?

$$\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}$$

Choose k out of n?

 $\frac{n!}{(n-k)! \times k!}$ 

Choose 2 out of n?

$$\frac{n\times(n-1)}{2}=\frac{n!}{(n-2)!\times 2}$$

Choose 3 out of n?

$$\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}$$

Choose k out of n?

 $\frac{n!}{(n-k)! \times k!}$ 

Choose 2 out of n?

$$\frac{n\times(n-1)}{2} = \frac{n!}{(n-2)!\times 2}$$

Choose 3 out of n?

$$\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}$$

Choose k out of n?

 $\frac{n!}{(n-k)! \times k!}$ 

Notation:  $\binom{n}{k}$  and pronounced "*n* choose *k*."
First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide...when possible.



3 card Poker deals: 52

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51$ 

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52\times51\times50$ 

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ .

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide...when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule.

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide...when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ?

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule.

Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ? Hand: Q, K, A.

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule.

Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ? Hand: Q, K, A. Deals: (Q, K, A),

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule.

Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ? Hand: Q, K, A. Deals: (Q, K, A), (Q, A, K),

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule.

Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ? Hand: Q, K, A. Deals: (Q, K, A), (Q, A, K), (K, A, Q), (K, A, Q), (A, K, Q), (A, Q, K).

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule.

Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ? Hand: Q, K, A.

Deals: (Q, K, A), (Q, A, K), (K, A, Q), (K, A, Q), (A, K, Q), (A, Q, K).  $\Delta = 3 \times 2 \times 1$ 

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule.

Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ?

Hand: Q, K, A.

Deals: (Q, K, A), (Q, A, K), (K, A, Q), (K, A, Q), (A, K, Q), (A, Q, K).

 $\Delta = 3 \times 2 \times 1$  First rule again.

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule.

Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ?

Hand: Q, K, A.

Deals: (Q, K, A), (Q, A, K), (K, A, Q), (K, A, Q), (A, K, Q), (A, Q, K).

 $\Delta = 3 \times 2 \times 1$  First rule again.

Total:

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule.

Second rule: when order doesn't matter divide...when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ?

Hand: Q, K, A.

Deals: (Q, K, A), (Q, A, K), (K, A, Q), (K, A, Q), (A, K, Q), (A, Q, K).

 $\Delta = 3 \times 2 \times 1$  First rule again. Total: 52!

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule.

Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ?

Hand: *Q*,*K*,*A*.

Deals: (Q,K,A), (Q,A,K), (K,A,Q), (K,A,Q), (A,K,Q), (A,Q,K).

 $\Delta = 3 \times 2 \times 1$  First rule again.

Total:  $\frac{52!}{49!3!}$  Second Rule!

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule.

Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ?

Hand: *Q*,*K*,*A*.

Deals: (Q, K, A), (Q, A, K), (K, A, Q), (K, A, Q), (A, K, Q), (A, Q, K).

 $\Delta = 3 \times 2 \times 1$  First rule again.

Total:  $\frac{52!}{49!3!}$  Second Rule!

Choose k out of n.

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule.

Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ?

Hand: *Q*,*K*,*A*.

Deals: (Q,K,A), (Q,A,K), (K,A,Q), (K,A,Q), (A,K,Q), (A,Q,K).

 $\Delta = 3 \times 2 \times 1$  First rule again.

Total:  $\frac{52!}{49|3|}$  Second Rule!

Choose k out of n.

Ordered set:  $\frac{n!}{(n-k)!}$ 

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule.

Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ?

Hand: *Q*,*K*,*A*.

Deals: (Q,K,A), (Q,A,K), (K,A,Q), (K,A,Q), (A,K,Q), (A,Q,K).

 $\Delta = 3 \times 2 \times 1$  First rule again.

Total:  $\frac{52!}{49(3)}$  Second Rule!

Choose *k* out of *n*. Ordered set:  $\frac{n!}{(n-k)!}$ 

What is  $\Delta$ ?

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule.

Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ?

Hand: *Q*,*K*,*A*.

Deals: (Q,K,A), (Q,A,K), (K,A,Q), (K,A,Q), (A,K,Q), (A,Q,K).

 $\Delta = 3 \times 2 \times 1$  First rule again.

Total:  $\frac{52!}{49(3)}$  Second Rule!

Choose *k* out of *n*. Ordered set:  $\frac{n!}{(n-k)!}$ 

What is  $\Delta ? k!$ 

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule.

Second rule: when order doesn't matter divide..when possible.



3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ?

Hand: *Q*,*K*,*A*.

Deals: (Q, K, A), (Q, A, K), (K, A, Q), (K, A, Q), (A, K, Q), (A, Q, K).

 $\Delta = 3 \times 2 \times 1$  First rule again.

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3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ?

Hand: *Q*,*K*,*A*.

Deals: (Q,K,A), (Q,A,K), (K,A,Q), (K,A,Q), (A,K,Q), (A,Q,K).

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Total:  $\frac{52!}{49!3!}$  Second Rule!

Choose *k* out of *n*. Ordered set:  $\frac{n!}{(n-k)!}$ What is  $\Delta$ ? *k*! First rule again.  $\implies$  Total:  $\frac{n!}{(n-k)!k!}$ 

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3 card Poker deals:  $52 \times 51 \times 50 = \frac{52!}{49!}$ . First rule. Poker hands:  $\Delta$ ?

Hand: *Q*,*K*,*A*.

Deals: (Q,K,A), (Q,A,K), (K,A,Q), (K,A,Q), (A,K,Q), (A,Q,K).

 $\Delta = 3 \times 2 \times 1$  First rule again.

Total:  $\frac{52!}{49!3!}$  Second Rule!

Choose *k* out of *n*. Ordered set:  $\frac{n!}{(n-k)!}$ What is  $\Delta$ ? *k*! First rule again.  $\implies$  Total:  $\frac{n!}{(n-k)!k!}$  Second rule.

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



Orderings of ANAGRAM?

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



Orderings of ANAGRAM? Ordered Set: 7!

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



Orderings of ANAGRAM? Ordered Set: 7! First rule.

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



Orderings of ANAGRAM? Ordered Set: 7! First rule. A's are the same!

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



Orderings of ANAGRAM? Ordered Set: 7! First rule. A's are the same! What is  $\Delta$ ?

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



Orderings of ANAGRAM? Ordered Set: 7! First rule. A's are the same! What is  $\Delta$ ? ANAGRAM

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



Orderings of ANAGRAM? Ordered Set: 7! First rule. A's are the same! What is  $\Delta$ ? ANAGRAM A<sub>1</sub>NA<sub>2</sub>GRA<sub>3</sub>M

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



Orderings of ANAGRAM? Ordered Set: 7! First rule. A's are the same! What is  $\Delta$ ? ANAGRAM A<sub>1</sub>NA<sub>2</sub>GRA<sub>3</sub>M, A<sub>2</sub>NA<sub>1</sub>GRA<sub>3</sub>M,

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



Orderings of ANAGRAM? Ordered Set: 7! First rule. A's are the same! What is  $\Delta$ ? ANAGRAM A<sub>1</sub>NA<sub>2</sub>GRA<sub>3</sub>M, A<sub>2</sub>NA<sub>1</sub>GRA<sub>3</sub>M, ...

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



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First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



Orderings of ANAGRAM? Ordered Set: 7! First rule. A's are the same! What is  $\Delta$ ? ANAGRAM A<sub>1</sub>NA<sub>2</sub>GRA<sub>3</sub>M, A<sub>2</sub>NA<sub>1</sub>GRA<sub>3</sub>M, ...  $\Delta = 3 \times 2 \times 1 = 3!$ 

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



Orderings of ANAGRAM? Ordered Set: 7! First rule. A's are the same! What is  $\Delta$ ? ANAGRAM A<sub>1</sub>NA<sub>2</sub>GRA<sub>3</sub>M, A<sub>2</sub>NA<sub>1</sub>GRA<sub>3</sub>M, ...  $\Delta = 3 \times 2 \times 1 = 3!$  First rule!
#### Example: Anagram

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



Orderings of ANAGRAM? Ordered Set: 7! First rule. A's are the same! What is  $\Delta$ ? ANAGRAM A<sub>1</sub>NA<sub>2</sub>GRA<sub>3</sub>M, A<sub>2</sub>NA<sub>1</sub>GRA<sub>3</sub>M, ...  $\Delta = 3 \times 2 \times 1 = 3!$  First rule!  $\Rightarrow \frac{7!}{3!}$ 

#### Example: Anagram

First rule:  $n_1 \times n_2 \cdots \times n_3$ . Product Rule. Second rule: when order doesn't matter divide..when possible.



Orderings of ANAGRAM? Ordered Set: 7! First rule. A's are the same! What is  $\Delta$ ? ANAGRAM A<sub>1</sub>NA<sub>2</sub>GRA<sub>3</sub>M, A<sub>2</sub>NA<sub>1</sub>GRA<sub>3</sub>M, ...  $\Delta = 3 \times 2 \times 1 = 3!$  First rule!  $\implies \frac{7!}{3!}$  Second rule!

How many orderings of letters of CAT?

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies 3 \times 2 \times 1$ 

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies$  3 × 2 × 1 = 3! orderings

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How many orderings of the letters in ANAGRAM?

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies$  3 × 2 × 1 = 3! orderings

How many orderings of the letters in ANAGRAM?

Ordered,

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies$  3 × 2 × 1 = 3! orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies$  3 × 2 × 1 = 3! orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A

total orderings of 7 letters.

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies$  3 × 2 × 1 = 3! orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A

total orderings of 7 letters. 7!

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies$  3  $\times$  2  $\times$  1 = 3! orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A

total orderings of 7 letters. 7! total "extra counts" or orderings of three A's?

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies$  3  $\times$  2  $\times$  1 = 3! orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A

total orderings of 7 letters. 7! total "extra counts" or orderings of three A's? 3!

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies$  3 × 2 × 1 = 3! orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A

total orderings of 7 letters. 7! total "extra counts" or orderings of three A's? 3!

Total orderings?

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

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How many orderings of the letters in ANAGRAM?

Ordered, except for A

total orderings of 7 letters. 7! total "extra counts" or orderings of three A's? 3!

Total orderings?  $\frac{7!}{3!}$ 

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies$  3 × 2 × 1 = 3! orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A

total orderings of 7 letters. 7! total "extra counts" or orderings of three A's? 3!

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Total orderings? \frac{7!}{3!}
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How many orderings of MISSISSIPPI?

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies$  3 × 2 × 1 = 3! orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A

total orderings of 7 letters. 7!

total "extra counts" or orderings of three A's? 3!

Total orderings?  $\frac{7!}{3!}$ 

How many orderings of MISSISSIPPI?

4 S's, 4 I's, 2 P's.

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies$  3 × 2 × 1 = 3! orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A

total orderings of 7 letters. 7!

total "extra counts" or orderings of three A's? 3!

Total orderings?  $\frac{7!}{3!}$ 

How many orderings of MISSISSIPPI?

4 S's, 4 I's, 2 P's. 11 letters total!

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies$  3 × 2 × 1 = 3! orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A

total orderings of 7 letters. 7!

total "extra counts" or orderings of three A's? 3!

Total orderings?  $\frac{7!}{3!}$ 

How many orderings of MISSISSIPPI?

4 S's, 4 I's, 2 P's.

11 letters total!

11! ordered objects!

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies$  3 × 2 × 1 = 3! orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A

total orderings of 7 letters. 7!

total "extra counts" or orderings of three A's? 3!

Total orderings?  $\frac{7!}{3!}$ 

How many orderings of MISSISSIPPI?

4 S's, 4 I's, 2 P's.

11 letters total!

11! ordered objects!

 $4! \times 4! \times 2!$  ordered objects per "unordered object"

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

 $\implies$  3 × 2 × 1 = 3! orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A

total orderings of 7 letters. 7!

total "extra counts" or orderings of three A's? 3!

Total orderings?  $\frac{7!}{3!}$ 

How many orderings of MISSISSIPPI?

4 S's, 4 I's, 2 P's.

11 letters total!

11! ordered objects!

 $4! \times 4! \times 2!$  ordered objects per "unordered object"

 $\implies \frac{11!}{4!4!2!}.$ 

Sample k items out of n

Sample *k* items out of *n* Without replacement:

Sample *k* items out of *n* Without replacement: Order matters:

Sample *k* items out of *n* Without replacement: Order matters:  $n \times$ 

Sample k items out of n

Without replacement: Order matters:  $n \times n - 1 \times n - 2 \dots$ 

Sample k items out of n

Without replacement:

Order matters:  $n \times n - 1 \times n - 2 \dots \times n - k + 1$ 

Sample k items out of n

Without replacement:

Order matters:  $n \times n - 1 \times n - 2 \dots \times n - k + 1 = \frac{n!}{(n-k)!}$ Order does not matter:

Sample k items out of n

Without replacement:

Order matters:  $n \times n - 1 \times n - 2 \dots \times n - k + 1 = \frac{n!}{(n-k)!}$ 

Order does not matter:

Second Rule: divide by number of orders

Sample k items out of n

Without replacement:

Order matters:  $n \times n - 1 \times n - 2 \dots \times n - k + 1 = \frac{n!}{(n-k)!}$ 

Order does not matter:

Second Rule: divide by number of orders – "k!"

Sample k items out of n

Without replacement:

Order matters:  $n \times n - 1 \times n - 2 \dots \times n - k + 1 = \frac{n!}{(n-k)!}$ Order does not matter: Second Rule: divide by number of orders – "*k*!"

 $\implies \frac{n!}{(n-k)!k!}$ , "*n* choose k" =  $\binom{n}{k}$ .

Sample k items out of n

Without replacement:

Order matters:  $n \times n - 1 \times n - 2 \dots \times n - k + 1 = \frac{n!}{(n-k)!}$ Order does not matter: Second Rule: divide by number of orders – "k!"  $\implies \frac{n!}{(n-k)!k!}$ , "n choose k" =  $\binom{n}{k}$ .

With Replacement.

Sample k items out of n

Without replacement:

Order matters:  $n \times n - 1 \times n - 2 \dots \times n - k + 1 = \frac{n!}{(n-k)!}$ Order does not matter:

Second Rule: divide by number of orders - "k!"

 $\implies \frac{n!}{(n-k)!k!}$ , "*n* choose *k*" =  $\binom{n}{k}$ .

With Replacement.

Order matters: n

Sample k items out of n

Without replacement:

Order matters:  $n \times n - 1 \times n - 2 \dots \times n - k + 1 = \frac{n!}{(n-k)!}$ Order does not matter:

Second Rule: divide by number of orders – "k!"

 $\implies \frac{n!}{(n-k)!k!}$ , "*n* choose *k*" =  $\binom{n}{k}$ .

With Replacement.

Order matters:  $n \times n$ 

Sample k items out of n

Without replacement:

Order matters:  $n \times n - 1 \times n - 2 \dots \times n - k + 1 = \frac{n!}{(n-k)!}$ Order does not matter:

Second Rule: divide by number of orders - "k!"

 $\implies \frac{n!}{(n-k)!k!}$ , "*n* choose *k*" =  $\binom{n}{k}$ .

With Replacement.

Order matters:  $n \times n \times \ldots n$ 

Sample k items out of n

Without replacement:

Order matters:  $n \times n - 1 \times n - 2 \dots \times n - k + 1 = \frac{n!}{(n-k)!}$ Order does not matter:

Second Rule: divide by number of orders - "k!"

 $\implies \frac{n!}{(n-k)!k!}$ , "*n* choose k" =  $\binom{n}{k}$ .

With Replacement.

Order matters:  $n \times n \times \ldots n = n^k$
Sample k items out of n

Without replacement:

Order matters:  $n \times n - 1 \times n - 2 \dots \times n - k + 1 = \frac{n!}{(n-k)!}$ Order does not matter:

Second Rule: divide by number of orders - "k!"

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Unordered elt: 1,2,2

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Unordered elt: 1,2,2 \frac{3!}{2!} ordered elts map to it.
```

How do we deal with this mess?!?!

## What we've learned so far

#### Sample *k* items out of *n*.

	With Replacement	Without Replacement
Order matters	n <sup>k</sup>	$\frac{n!}{(n-k)!}$
Order doesn't matter	????	$\binom{n}{k}$



Short break.

There are 5 people in a room.

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There are 5 people in a room. They all have different heights. *i* gives a handshake to *j*, if only if *j* is shorter than *i*. How many handshakes?

How many ways can Bob and Alice split 5 dollars?

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5 dollars for Bob and 0 for Alice:

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5 dollars for Bob and 0 for Alice: one ordered set: (B, B, B, B, B).

How many ways can Bob and Alice split 5 dollars? For each of 5 dollars pick Bob or Alice(2<sup>5</sup>), divide out order ???

5 dollars for Bob and 0 for Alice: one ordered set: (B, B, B, B, B).

4 for Bob and 1 for Alice:

How many ways can Bob and Alice split 5 dollars? For each of 5 dollars pick Bob or Alice(2<sup>5</sup>), divide out order ???

5 dollars for Bob and 0 for Alice: one ordered set: (B, B, B, B, B).

4 for Bob and 1 for Alice:

5 ordered sets: (*A*, *B*, *B*, *B*, *B*); (*B*, *A*, *B*, *B*, *B*); ...

How many ways can Bob and Alice split 5 dollars? For each of 5 dollars pick Bob or Alice(2<sup>5</sup>), divide out order ???

5 dollars for Bob and 0 for Alice: one ordered set: (B, B, B, B, B).

4 for Bob and 1 for Alice: 5 ordered sets: (*A*, *B*, *B*, *B*, *B*); (*B*, *A*, *B*, *B*, *B*); ...

"Sorted" way to specify, first Alice's dollars, then Bob's.

(B, B, B, B, B)(A, B, B, B, B)(A, A, B, B, B)

How many ways can Bob and Alice split 5 dollars? For each of 5 dollars pick Bob or Alice(2<sup>5</sup>), divide out order ???

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"Sorted" way to specify, first Alice's dollars, then Bob's.

(B, B, B, B, B, B)(A, B, B, B, B)(A, A, B, B, B)and so on.

How many ways can Bob and Alice split 5 dollars? For each of 5 dollars pick Bob or Alice(2<sup>5</sup>), divide out order ???

5 dollars for Bob and 0 for Alice: one ordered set: (B, B, B, B, B).

4 for Bob and 1 for Alice: 5 ordered sets: (*A*, *B*, *B*, *B*, *B*); (*B*, *A*, *B*, *B*, *B*); ...

"Sorted" way to specify, first Alice's dollars, then Bob's.

(B, B, B, B, B, B)(A, B, B, B, B)(A, A, B, B, B)and so on.



Second rule of counting is no good here!

# Splitting 5 dollars.

How many ways can Bob and Alice split 5 dollars?

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How many ways can Bob and Alice split 5 dollars? Well, I can actually do this by bruteforcing....

How many ways can Bob and Alice split 5 dollars? Well, I can actually do this by bruteforcing.... 0\$ to Alice. How many ways can Bob and Alice split 5 dollars? Well, I can actually do this by bruteforcing.... 0\$ to Alice. or 1\$ to Alice. How many ways can Bob and Alice split 5 dollars? Well, I can actually do this by bruteforcing.... 0\$ to Alice. or 1\$ to Alice. or 2\$ to Alice.

# Splitting 5 dollars.

How many ways can Bob and Alice split 5 dollars? Well, I can actually do this by bruteforcing.... 0\$ to Alice. or 1\$ to Alice. or 2\$ to Alice. or 3\$ to Alice. or 4\$ to Alice. or 5\$ to Alice. How many ways can Bob and Alice split 5 dollars? Well, I can actually do this by bruteforcing.... 0\$ to Alice. or 1\$ to Alice. or 2\$ to Alice. or 3\$ to Alice. or 4\$ to Alice. or 5\$ to Alice. How do we generalize?
How many ways can Alice, Bob, and Eve split 5 dollars.

How many ways can Alice, Bob, and Eve split 5 dollars. Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).

How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).

Separate Alice's dollars from Bob's and then Bob's from Eve's.

How many ways can Alice, Bob, and Eve split 5 dollars.

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Five dollars are five stars: \*\*\*\*.

How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).

Separate Alice's dollars from Bob's and then Bob's from Eve's.

Five dollars are five stars: \*\*\*\*.

Alice: 2, Bob: 1, Eve: 2.

How many ways can Alice, Bob, and Eve split 5 dollars.

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Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).
```

Separate Alice's dollars from Bob's and then Bob's from Eve's.

Five dollars are five stars: \*\*\*\*.

Alice: 2, Bob: 1, Eve: 2. Stars and Bars:  $\star \star |\star| \star \star$ .

How many ways can Alice, Bob, and Eve split 5 dollars.

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Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).
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Separate Alice's dollars from Bob's and then Bob's from Eve's.

Five dollars are five stars: \*\*\*\*.

Alice: 2, Bob: 1, Eve: 2.

Stars and Bars:  $\star \star |\star| \star \star$ .

Alice: 0, Bob: 1, Eve: 4.

How many ways can Alice, Bob, and Eve split 5 dollars.

```
Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).
```

Separate Alice's dollars from Bob's and then Bob's from Eve's.

Five dollars are five stars: \*\*\*\*.

Alice: 2, Bob: 1, Eve: 2. Stars and Bars:  $\star \star |\star| \star \star$ .

Alice: 0, Bob: 1, Eve: 4. Stars and Bars:  $|\star| \star \star \star \star$ .

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Each split "is" a sequence of stars and bars.

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Alice: 0, Bob: 1, Eve: 4. Stars and Bars: |\*|\*\*\*\*.

Each split "is" a sequence of stars and bars. Each sequence of stars and bars "is" a split.

Counting Rule: if there is a one-to-one mapping between two sets they have the same size!

How many different 5 star and 2 bar diagrams?

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| \* | \* \* \* \*.

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\* \* \* \* \* \*.

7 positions in which to place the 2 bars.

\_\_\_\_\_

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\* \* \* \* \* \*.

\_ \_ \_ \_ \_ \_ \_

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4

How many different 5 star and 2 bar diagrams?

\* \* \* \* \* \*.

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7 positions in which to place the 2 bars.

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Alice: 0; Bob 1; Eve: 4
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How many different 5 star and 2 bar diagrams?

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7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4  $| \star | \star \star \star \star$ . Bars in first and third position.

How many different 5 star and 2 bar diagrams?

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\_ \_ \_ \_ \_ \_ \_

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4  $| \star | \star \star \star \star$ . Bars in first and third position. Alice: 1; Bob 4; Eve: 0

How many different 5 star and 2 bar diagrams?

\* \* \* \* \* \*.

\_ \_ \_ \_ \_ \_ \_

7 positions in which to place the 2 bars.

```
Alice: 0; Bob 1; Eve: 4
| \star | \star \star \star.
Bars in first and third position.
Alice: 1; Bob 4; Eve: 0
\star | \star \star \star \star |.
```

How many different 5 star and 2 bar diagrams?

\* \* \* \* \* \*.

\_ \_ \_ \_ \_ \_

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4  $| \star | \star \star \star$ . Bars in first and third position. Alice: 1; Bob 4; Eve: 0  $\star | \star \star \star \star |$ . Bars in second and seventh position.

How many different 5 star and 2 bar diagrams?

| \* | \* \* \* \*.

\_ \_ \_ \_ \_ \_

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4  $| \star | \star \star \star \star$ . Bars in first and third position. Alice: 1; Bob 4; Eve: 0  $\star | \star \star \star \star |$ . Bars in second and seventh position.  $\binom{7}{2}$  ways to do so and

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7 positions in which to place the 2 bars.

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Alice: 0; Bob 1; Eve: 4
| \star | \star \star \star \star.
Bars in first and third position.
```

```
Alice: 1; Bob 4; Eve: 0
```

\* | \* \* \* \* |.

Bars in second and seventh position.

 $\binom{7}{2}$  ways to do so and

 $\binom{7}{2}$  ways to split 5 dollars among 3 people.

Ways to add up *n* (non-negative) numbers to sum to *k*?

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In general, k stars n-1 bars.

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n+k-1 positions from which to choose n-1 bar positions.

```
\binom{n+k-1}{n-1}
```

Or: *k* unordered choices from set of *n* possibilities with replacement. **Sample with replacement where order doesn't matter.** 

**First rule:**  $n_1 \times n_2 \cdots \times n_3$ .

**First rule:**  $n_1 \times n_2 \cdots \times n_3$ .

k Samples with replacement from *n* items:  $n^k$ .

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```
k Samples with replacement from n items: n^k.
Sample without replacement: \frac{n!}{(n-k)!}
```

**First rule:**  $n_1 \times n_2 \cdots \times n_3$ .

*k* Samples with replacement from *n* items:  $n^k$ . Sample without replacement:  $\frac{n!}{(n-k)!}$ 

Second rule: when order doesn't matter (sometimes) can divide...

**First rule:**  $n_1 \times n_2 \cdots \times n_3$ .

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# Second rule: when order doesn't matter (sometimes) can divide...

Sample without replacement and order doesn't matter:  $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ . "*n* choose *k*"

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**One-to-one rule: equal in number if one-to-one correspondence.** pause Bijection!
## Summary.

**First rule:**  $n_1 \times n_2 \cdots \times n_3$ .

*k* Samples with replacement from *n* items:  $n^k$ . Sample without replacement:  $\frac{n!}{(n-k)!}$ 

## Second rule: when order doesn't matter (sometimes) can divide...

Sample without replacement and order doesn't matter:  $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ . "*n* choose *k*"

**One-to-one rule: equal in number if one-to-one correspondence.** pause Bijection!

Sample *k* times from *n* objects with replacement and order doesn't matter:  $\binom{k+n-1}{n-1}$ .

## What we've learned so far

## Sample *k* items out of *n*.

	With Replacement	Without Replacement
Order matters	n <sup>k</sup>	$\frac{n!}{(n-k)!}$
Order doesn't matter	$\binom{n+k-1}{n-1}$	$\binom{n}{k}$