CS70: Counting

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Reminder: Don't write on the board.

Lecture 9

What's to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?
$\frac{3}{8}$. How did I know?

Count blue. Count total. Divide.

Today (and tomorrow): Counting!

Next week: Probability.

Make sure you understand counting if you want to understand probability!!!

Outline: basics

1. Counting.
2. Rules of Counting.
3. Sample with/without replacement where order does/doesn't matter.
4. Combinatorial proofs (mostly tomorrow)

Count?

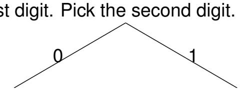
$1+1=?$ 2
$3+4=?$ 7
How many 100-bit strings are there that contain exactly 6 ones?
$1,192,052,400$

Count?

How many outcomes possible for k coin tosses?
How many poker hands?
How many handshakes for n people?
How many 10 digit numbers?
How many 10 digit numbers without repetition?

Using a tree.

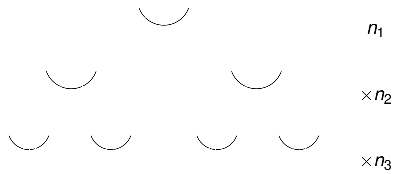
How many 3-bit strings? (I know, I know... Calm down....)
How many different sequences of three bits from $\{0, 1\}$?
How would you make one sequence?
Pick the first digit. Pick the second digit. Pick the third digit.



8 leaves which is $2 \times 2 \times 2$. One leaf for each string.
8 3-bit strings!

First Rule of Counting: Product Rule

Objects made by choosing from n_1 options, then n_2 options , ..., then n_k options: the number of objects is $n_1 \times n_2 \cdots \times n_k$.



In picture, $2 \times 2 \times 3 = 12$

Using the first rule.

How many outcomes possible for k coin tosses?

2 ways for first choice, 2 ways for second choice, ...
$2 \times 2 \cdots \times 2 = 2^k$

How many k digit numbers (in decimal)?

10 ways for first choice, 10 ways for second choice, ...
$10 \times 10 \cdots \times 10 = 10^k$

How many n digit base m numbers?

m ways for first, m ways for second, ...
m^n

Functions, polynomials.

How many functions f mapping $S = \{s_1, s_2, \ldots\}$ to $T = \{t_1, t_2, \ldots\}$?

$|T|$ ways to choose for $f(s_1)$, $|T|$ ways to choose for $f(s_2)$, ...

....$|T|^{|S|}$

How many polynomials of degree d, when the coefficients of the polynomial come from the set $\{0, 1, \ldots, p-1\}$?

p ways to choose for first coefficient, p ways for second, ...
...p^{d+1}

Permutations.

How many 10 digit numbers? 10^{10}.

How many 10 digit numbers **without repeating a digit**?

10 ways for first, 9 ways for second, 8 ways for third, ...

... $10 * 9 * 8 \cdots * 1 = 10!$ [1]

How many orderings of n objects are there?
Permutations of n objects.

n ways for first, $n-1$ ways for second,
$n-2$ ways for third, ...

... $n * (n-1) * (n-2) \cdot * 1 = n!$

[1] By definition: 0! = 1.

One-to-One Functions.

How many one-to-one functions from S to S.

$|S|$ choices for $f(s_1)$, $|S|-1$ choices for $f(s_2)$, ...
So total number is $|S| \times (|S|-1) \cdots 1 = |S|!$
A one-to-one function is a permutation!

Counting sets when order doesn't matter.

How many poker hands? (5 cards)

$52 \times 51 \times 50 \times 49 \times 48$???

Aren't $A, K, Q, 10, J$ of spades
and $10, J, Q, K, A$ of spades the same?

Second Rule of Counting: If order doesn't matter count ordered objects and then divide by number of orderings.[2]

Number of orderings for a poker hand: 5!

$$\frac{52 \times 51 \times 50 \times 49 \times 48}{5!}$$

Can write as...

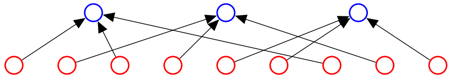
$$\frac{52!}{5! \times 47!}$$

Generic: ways to choose 5 out of 52 possibilities.

[2]When each unordered object corresponds to an equal numbers of ordered objects.

Ordered to unordered.

Second Rule of Counting: If order doesn't matter count ordered objects and then divide by number of orderings.



How many red nodes (ordered objects)? 9.

How many red nodes mapped to one blue node? 3.

How many blue nodes (unordered objects)? $\frac{9}{3} = 3$.

If you know:(1) how many red vertices and (2) in-degree for each blue vertex.

Then, you know how many blue vertices!

How many poker deals? (red vertices) $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$.

How many poker hands per deal? (degree) Map each deal to ordered deal. 5!

How many poker hands? $\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!}$

..order doesn't matter.

Choose 2 out of n?

$$\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}$$

Choose 3 out of n?

$$\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}$$

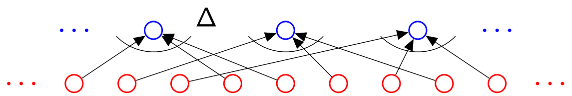
Choose k **out of** n?

$$\frac{n!}{(n-k)! \times k!}$$

Notation: $\binom{n}{k}$ **and pronounced "n choose k."**

Example: Visualize the proof..

First rule: $n_1 \times n_2 \cdots \times n_3$**. Product Rule.**
Second rule: when order doesn't matter divide..when possible.

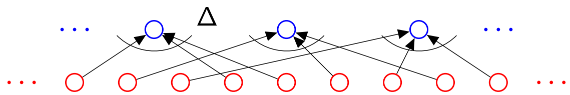


3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$. First rule.
Poker hands: Δ?
Hand: Q, K, A.
Deals: (Q,K,A), (Q,A,K), (K,A,Q),(K,A,Q), (A,K,Q), (A,Q,K).
Δ = $3 \times 2 \times 1$ First rule again.
Total: $\frac{52!}{49!3!}$ Second Rule!

Choose k out of n.
Ordered set: $\frac{n!}{(n-k)!}$
What is Δ? $k!$ First rule again.
$\implies$ Total: $\frac{n!}{(n-k)!k!}$ Second rule.

Example: Anagram

First rule: $n_1 \times n_2 \cdots \times n_3$**. Product Rule.**
Second rule: when order doesn't matter divide..when possible.



Orderings of ANAGRAM?
Ordered Set: 7! First rule.
A's are the same!
What is Δ?
ANAGRAM
$A_1NA_2GRA_3M$, $A_2NA_1GRA_3M$, ...
$\Delta = 3 \times 2 \times 1 = 3!$ First rule!
$\implies \frac{7!}{3!}$ Second rule!

Some Practice.

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.

$\implies 3 \times 2 \times 1 = 3!$ orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A

total orderings of 7 letters. 7!
total "extra counts" or orderings of three A's? 3!

Total orderings? $\frac{7!}{3!}$

How many orderings of MISSISSIPPI?

4 S's, 4 I's, 2 P's.
11 letters total!
11! ordered objects!
$4! \times 4! \times 2!$ ordered objects per "unordered object"
$\implies \frac{11!}{4!4!2!}$.

Sampling...

Sample k items out of n

Without replacement:
Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$
Order does not matter:
Second Rule: divide by number of orders – “k!”
$\implies \frac{n!}{(n-k)!k!}$, “$n$ choose k” = $\binom{n}{k}$.

With Replacement.
Order matters: $n \times n \times \ldots n = n^k$
Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!
So different number of unordered elements map to each unordered element!

Unordered elt: 1,2,3 $\quad$ 3! ordered elts map to it.
Unordered elt: 1,2,2 $\quad \frac{3!}{2!}$ ordered elts map to it.

How do we deal with this mess?!?!

What we’ve learned so far

Sample k items out of n.

	With Replacement	Without Replacement
Order matters	n^k	$\frac{n!}{(n-k)!}$
Order doesn’t matter	????	$\binom{n}{k}$

Break

Short break.

Sanity check

There are 5 people in a room.
They all have different heights.
i gives a handshake to j, if only if j is shorter than i.
How many handshakes?

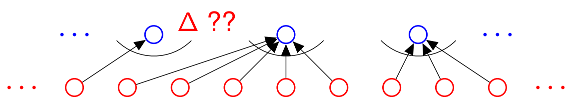
Splitting up some money....

How many ways can Bob and Alice split 5 dollars?
For each of 5 dollars pick Bob or Alice(2^5), divide out order ???

5 dollars for Bob and 0 for Alice:
one ordered set: (B,B,B,B,B).

4 for Bob and 1 for Alice:
5 ordered sets: (A,B,B,B,B) ; (B,A,B,B,B); ...

“Sorted” way to specify, first Alice’s dollars, then Bob’s.
(B,B,B,B,B) (B,B,B,B,B)
(A,B,B,B,B) (A,B,B,B,B),(B,A,B,B,B),(B,B,A,B,B),...
(A,A,B,B,B) (A,A,B,B,B),(A,B,A,B,B),(A,B,B,A,B),...
and so on.



Second rule of counting is no good here!

Splitting 5 dollars.

How many ways can Bob and Alice split 5 dollars?
Well, I can actually do this by bruteforcing....
0$ to Alice.
or 1$ to Alice.
or 2$ to Alice.
or 3$ to Alice.
or 4$ to Alice.
or 5$ to Alice.
How do we generalize?

Splitting 5 dollars.

How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).

Separate Alice's dollars from Bob's and then Bob's from Eve's.

Five dollars are five stars: $\star\star\star\star\star$.

Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: $\star\star|\star|\star\star$.

Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: $|\star|\star\star\star\star$.

Each split "is" a sequence of stars and bars.
Each sequence of stars and bars "is" a split.

Counting Rule: if there is a one-to-one mapping between two sets they have the same size!

Stars and Bars.

How many different 5 star and 2 bar diagrams?

$|\star|\star\star\star\star$.

7 positions in which to place the 2 bars.

$- - - - - - -$

Alice: 0; Bob 1; Eve: 4
$|\star|\star\star\star\star$.
Bars in first and third position.

Alice: 1; Bob 4; Eve: 0
$\star|\star\star\star\star|$.
Bars in second and seventh position.

$\binom{7}{2}$ ways to do so and
$\binom{7}{2}$ ways to split 5 dollars among 3 people.

Stars and Bars.

Ways to add up n (non-negative) numbers to sum to k?
(For example, how many ways to add up 10 numbers to sum to 50?)
"Sampling with replacement where order doesn't matter."

In general, k stars $n-1$ bars.

$$\star\star|\star|\cdots|\star\star.$$

$n+k-1$ positions from which to choose $n-1$ bar positions.

$$\binom{n+k-1}{n-1}$$

Or: k unordered choices from set of n possibilities with replacement.
Sample with replacement where order doesn't matter.

Summary.

First rule: $n_1 \times n_2 \cdots \times n_3$.

k Samples with replacement from n items: n^k.
Sample without replacement: $\frac{n!}{(n-k)!}$

Second rule: when order doesn't matter (sometimes) can divide...

Sample without replacement and order doesn't matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
"n choose k"

One-to-one rule: equal in number if one-to-one correspondence.
pause Bijection!

Sample k times from n objects with replacement and order doesn't matter: $\binom{k+n-1}{n-1}$.

What we've learned so far

Sample k items out of n.

	With Replacement	Without Replacement
Order matters	n^k	$\frac{n!}{(n-k)!}$
Order doesn't matter	$\binom{n+k-1}{n-1}$	$\binom{n}{k}$