CS70: Discrete Math and Probability

Slides adopted from Satish Rao, CS70 Spring 2016

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Introduction

Programming Computers $\equiv$ Superpower!

What are your super powerful programs doing?
Logic and Proofs!
Induction $\equiv$ Recursion.

What can computers do?
Work with discrete objects.
Discrete Math $\implies$ immense application.

Computers learn and interact with the world?
E.g. machine learning, data analysis.
Probability!

See note 1, for more discussion.

Admin.

Course Webpage: **www.eecs70.org**

Explains policies, has homework/discussion worksheets, slides, exam dates, etc.

Questions $\Longrightarrow$ piazza:
piazza.com/berkeley/summer2016/cs70

Assessment:
- Homework: 20%
- Midterm 1 (07/08): 20%
- Midterm 2 (07/29): 20%
- Final (08/12): 35%
- Quiz: 4%
- Sundry: 1%

Conflicts? Piazza pinned post.

Learning tips

Take homework seriously!
Go to homework parties, study groups
VERY fast paced, start early
Use piazza, help each other out

Questions?

Staffs

3 Co-Instructors

Fan Ye

Just graduated, from Berkeley

Been TA for CS70 for two semesters

Will start working at Google as a software engineer on September

Enjoy climbing, badminton, boxing, also like to watch movies and games
Recently I'm climbing . . . the ladder of league of legends ranking system . . .

Office hours: Monday 10-11, Tuesday 11-12 in Soda 611 or by appointment

Grace Dinh

Email: `dinh@berkeley.edu`
Office Hours: M/W 3:30-5:00 (right after lecture) in 606 Soda
I just finished my first year of grad school. My research interests are numerical algorithms and complexity theory - essentially, I work on making faster algorithms for doing things like solving equations, factoring matrices, etc. (and proving that they run fast!), as well as showing that there are limits on how fast we can make these algorithms.
Also did my undergrad here at Cal - CS70 was by far my favorite lower-div.
Fun fact: I like to make ice cream.

Alex Psomas

Not here today. Tomorrow lecture

Staffs

3 Co-Instructors
12 awesome and talented TAs.

Wason's experiment:1

Suppose we have four cards on a table:

- 1st about Alice, 2nd about Bob, 3rd Charlie, 4th Donna.
- Card contains person's destination on one side,
  and mode of travel.
- Consider the theory:
  "If a person travels to Chicago, he/she flies."
- Suppose you see that Alice went to Baltimore, Bob drove, Charlie went to Chicago, and Donna flew.

Alice	Bob	Charlie	Donna
Baltimore	drove	Chicago	flew

- Which cards do you need to flip to test the theory?

Answer: Later.

CS70: Lecture 1. Outline.

Today: Note 1.   Note 0 is background. Do read/skim it.

The language of proofs!

1. Propositions.
2. Propositional Forms.
3. Implication.
4. Truth Tables
5. Quantifiers
6. More De Morgan's Laws

Propositions: Statements that are true or false.

$\sqrt{2}$ is irrational	**Proposition**	True
2+2 = 4	**Proposition**	True
2+2 = 3	**Proposition**	False
826th digit of pi is 4	**Proposition**	False
Stephen Curry is a good player.	**Not a Proposition**	
All evens > 2 are sums of 2 primes	**Proposition**	False
$4+5$	**Not a Proposition.**	
$x+x$	**Not a Proposition.**	
Alice travelled to Chicago	**Proposition.**	False

Again: "value" of a proposition is ... True or False

Propositional Forms.

Put propositions together to make another...

Conjunction ("and"): $P \wedge Q$

"$P \wedge Q$" is True when both P and Q are True . Else False .

Disjunction ("or"): $P \vee Q$

"$P \vee Q$" is True when at least one P or Q is True . Else False .

Negation ("not"): $\neg P$

"$\neg P$" is True when P is False . Else False .

Examples:

$\neg$ "$(2+2=4)$"	– a proposition that is ... False
"$2+2=3$" $\bigwedge$ "$2+2=4$"	– a proposition that is ... False
"$2+2=3$" $\bigvee$ "$2+2=4$"	– a proposition that is ... True

Propositional Forms: quick check!

$P =$ "$\sqrt{2}$ is rational"
$Q =$ "826th digit of pi is 2"

P is ...False .
Q is ...True .

$P \wedge Q$ ... False

$P \vee Q$ ... True

$\neg P$ ... True

Put them together..

Propositions:

C_1 - Take class 1
C_2 - Take class 2
....

You can only take one of class 1, class 2 and one of class 3, class 4, or take both class 2, class 3 and take either class 5 or class 6, as long as you are not taking class 4 at the same time.

Propositional Form:

$$((C_1 \vee C_2) \wedge (C_3 \vee C_4)) \vee ((C_2 \wedge C_3) \wedge (C_5 \vee C_6) \wedge (\neg C_4))$$

Can you take class 1?
Can you take class 1 and class 5 together?

This seems ...complicated.

We can program!!!!We need a way to keep track!

Truth Tables for Propositional Forms.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Notice: $\wedge$ and $\vee$ are commutative.

One use for truth tables: Logical Equivalence of propositional forms!

Example: $\neg(P \wedge Q)$ logically equivalent to $\neg P \vee \neg Q$

...because the two propositional forms have the same...

....Truth Table!

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

DeMorgan's Law's for Negation: distribute and flip!

$\neg(P \wedge Q) \quad \equiv \quad \neg P \vee \neg Q$ $\qquad\qquad$ $\neg(P \vee Q) \quad \equiv \quad \neg P \wedge \neg Q$

Distributive?

$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$?

Simplify: $(T \wedge Q) \equiv Q$, $(F \wedge Q) \equiv F$.

Cases:

P is True .

LHS: $T \wedge (Q \vee R) \equiv (Q \vee R)$.

RHS: $(T \wedge Q) \vee (T \wedge R) \equiv (Q \vee R)$.

P is False .

LHS: $F \wedge (Q \vee R) \equiv F$.

RHS: $(F \wedge Q) \vee (F \wedge R) \equiv (F \vee F) \equiv F$.

$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$?

Simplify: $T \vee Q \equiv T$, $F \vee Q \equiv Q$.

Foil 1:

$(A \vee B) \wedge (C \vee D) \equiv (A \wedge C) \vee (A \wedge D) \vee (B \wedge C) \vee (B \wedge D)$?

Foil 2:

$(A \wedge B) \vee (C \wedge D) \equiv (A \vee C) \wedge (A \vee D) \wedge (B \vee C) \wedge (B \vee D)$?

Implication.

$P \implies Q$ interpreted as

If P, then Q.

True Statements: P, $P \implies Q$.
Conclude: Q is true.

Examples:

Statement: If you stand in the rain, then you'll get wet.
P = "you stand in the rain"
Q = "you will get wet"
Statement: "Stand in the rain"

Can conclude: "you'll get wet."

Statement: If a right triangle has sidelengths $a \leq b \leq c$, then $a^2 + b^2 = c^2$.

P = "a right triangle has sidelengths $a \leq b \leq c$",
Q = "$a^2 + b^2 = c^2$".

Non-Consequences/consequences of Implication

The statement "$P \implies Q$"

only is False if P is True and Q is False .

False implies nothing
P False means Q can be True or False
Anything implies true.
P can be True or False when Q is True

If chemical plant pollutes river, fish die.
If fish die, did chemical plant pollute river?

Not necessarily.

$P \implies Q$ and Q are True does not mean P is True

Be careful!

Instead we have:
$P \implies Q$ and P are True *does* mean Q is True .

The chemical plant pollutes river. Can we conclude fish die?

Some Fun: use propositional formulas to describe implication?
$((P \implies Q) \wedge P) \implies Q.$

Implication and English.

$P \implies Q$

- If P, then Q.
- Q if P.
  Just reversing the order.
- P only if Q.
  Remember if P is true then Q must be true.
  this suggests that P can only be true if Q is true.
  since if Q is false P must have been false.
- P is sufficient for Q.
  This means that proving P allows you
  to conclude that Q is true.
- Q is necessary for P.
  For P to be true it is necessary that Q is true.
  Or if Q is false then we know that P is false.

Truth Table: implication.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$\neg P \vee Q$
T	T	T
T	F	F
F	T	T
F	F	T

$\neg P \vee Q \quad \equiv P \implies Q.$

These two propositional forms are logically equivalent!

Contrapositive, Converse

- Contrapositive of $P \implies Q$ is $\neg Q \implies \neg P$.
  - If the plant pollutes, fish die.
  - If the fish don't die, the plant does not pollute.
    (contrapositive)

  - If you stand in the rain, you get wet.
  - If you did not stand in the rain, you did not get wet.
    (not contrapositive!) converse!
  - If you did not get wet, you did not stand in the rain.
    (contrapositive.)

  Logically equivalent! Notation: $\equiv$.
  $P \implies Q \equiv \neg P \vee Q \equiv \neg(\neg Q) \vee \neg P \equiv \neg Q \implies \neg P$.
- Converse of $P \implies Q$ is $Q \implies P$.
  If fish die the plant pollutes.
  Not logically equivalent!
- **Definition:** If $P \implies Q$ and $Q \implies P$ is P if and only if Q or $P \iff Q$.
  (Logically Equivalent: $\iff$.)

Variables.

Propositions?

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
- $x > 2$
- n is even and the sum of two primes

No. They have a free variable.

We call them predicates, e.g., $Q(x) =$ "x is even"
Same as boolean valued functions from 61A or 61AS!

- $P(n) =$ "$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$."
- $R(x) =$ "$x > 2$"
- $G(n) =$ "n is even and the sum of two primes"
- Remember Wason's experiment!
  $F(x) =$ "Person x flew."
  $C(x) =$ "Person x went to Chicago
- $C(x) \implies F(x)$. Theory from Wason's.
  If person x goes to Chicago then person x flew.

Next: Statements about boolean valued functions!!

Quantifiers..

There exists quantifier:

$(\exists x \in S)(P(x))$ means "There exists an x in S where $P(x)$ is true."

For example:

$$(\exists x \in \mathbb{N})(x = x^2)$$

Equivalent to "$(0 = 0) \vee (1 = 1) \vee (2 = 4) \vee \ldots$"

Much shorter to use a quantifier!

For all quantifier;
$(\forall x \in S)\ (P(x))$. means "For all x in S, we have $P(x)$ is True ."

Examples:

"Adding 1 makes a bigger number." $(\forall x \in \mathbb{N})\ (x + 1 > x)$

""the square of a number is always non-negative" $(\forall x \in \mathbb{N})(x^2 >= 0)$

Wait! What is $\mathbb{N}$?

Quantifiers: universes.

Proposition: "For all natural numbers n**,** $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$**."**

Proposition has **universe**: "the natural numbers".

Universe examples include..

- $\mathbb{N} = \{0, 1, \ldots\}$ (natural numbers).
- $\mathbb{Z} = \{\ldots, -1, 0, \ldots\}$ (integers)
- $\mathbb{Z}^+$ (positive integers)
- $\mathbb{R}$ (real numbers)
- Any set: $S = \{Alice, Bob, Charlie, Donna\}$.
- See note 0 for more!

Back to: Wason's experiment:1

Theory:
"If a person travels to Chicago, he/she flies."

Suppose you see that Alice went to Baltimore, Bob drove, Charlie went to Chicago, and Donna flew.

Which cards do you need to flip to test the theory?

$P(x)$ = "Person x went to Chicago." $Q(x)$ = "Person x flew"

Statement/theory: $\forall x \in \{A, B, C, D\}, P(x) \implies Q(x)$

$P(A) =$ False . Do we care about $Q(A)$?
No. $P(A) \implies Q(A)$, when $P(A)$ is False , $Q(A)$ can be anything.
$Q(B) =$ False . Do we care about $P(B)$?
Yes. $P(B) \implies Q(B) \equiv \neg Q(B) \implies \neg P(B)$.
So $P(Bob)$ must be False .
$P(C) =$ True . Do we care about $P(C)$?
Yes. $P(C) \implies Q(C)$ means $Q(C)$ must be true.
$Q(D) =$ True . Do we care about $P(D)$?
No. $P(D) \implies Q(D)$ holds whatever $P(D)$ is when $Q(D)$ is true.
Only have to turn over cards for Bob and Charlie.

More for all quantifiers examples.

- “doubling a natural number always makes it larger”

$$(\forall x \in N)\ (2x > x) \quad \textbf{False Consider } x = 0$$

Can fix statement...

$$(\forall x \in N)\ (2x \geq x) \quad \textbf{True}$$

- “Square of any natural number greater than 5 is greater than 25.”

$$(\forall x \in N)(x > 5 \implies x^2 > 25).$$

Idea alert: Restrict domain using implication.

Note that we may omit universe if clear from context.

Quantifiers..not commutative.

- In English: “there is a natural number that is the square of every natural number”.

$$(\exists y \in N)\ (\forall x \in N)\ (y = x^2) \quad \text{False}$$

- In English: “the square of every natural number is a natural number.”

$$(\forall x \in N)(\exists y \in N)\ (y = x^2) \quad \text{True}$$

Quantifiers....negation...DeMorgan again.

Consider

$$\neg(\forall x \in S)(P(x)),$$

English: there is an x in S where $P(x)$ does not hold.

That is,

$$\neg(\forall x \in S)(P(x)) \iff \exists(x \in S)(\neg P(x)).$$

What we do in this course! We consider claims.

Claim: $(\forall x)\ P(x)$ "For all inputs x the program works."
For False , find x, where $\neg P(x)$.
Counterexample.
Bad input.
Case that illustrates bug.
For True : prove claim. Next lectures...

Negation of exists.

Consider

$$\neg(\exists x \in S)(P(x))$$

English: means that for all x in S , $P(x)$ does not hold.

That is,

$$\neg(\exists x \in S)(P(x)) \iff \forall(x \in S)\neg P(x).$$

Talk logically

Theorem: No three positive integers a, b, c satisfy the equation $a^n + b^n = c^b$ for any integer n strictly larger than two.

Which Theorem?

Fermat's Last Theorem!

How to express this theorem using propositions?

$(\forall n \in N); \neg(\exists a, b, c \in N);(n \geq 3 \implies a^n + b^n = c^n)$

Using implication to state edge case restrictions (for any integer strictly greater than two)

DeMorgan Restatement:
Theorem: $\neg(\exists n \in N)\ (\exists a, b, c \in N)\ (n \geq 3 \implies a^n + b^n = c^n)$

Summary.

Propositions are statements that are true or false.

Proprositional forms use $\wedge, \vee, \neg$.

Propositional forms correspond to truth tables.

Logical equivalence of forms means same truth tables.

Implication: $P \implies Q \iff \neg P \vee Q$.

Contrapositive: $\neg Q \implies \neg P$
Converse: $Q \implies P$

Predicates: Statements with "free" variables.

Quantifiers: $\forall x\ P(x)$, $\exists y\ Q(y)$

Now can state theorems! And disprove false ones!

DeMorgans Laws: "Flip and Distribute negation"

$$\neg(P \vee Q) \iff (\neg P \wedge \neg Q)$$
$$\neg \forall x\ P(x) \iff \exists x\ \neg P(x).$$
$$\neg \exists x\ P(x) \iff \forall x\ \neg P(x).$$

Next Time: proofs!