CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Dinh, Psomas, and Ye HW 4

Due Monday July 18 at 1:59PM

1. **Sample Space and Events (1/1/1/1/1/1/2/2 points)**
   Consider the sample space Ω of all outcomes from flipping a coin 3 times.

   (a) List all the outcomes in Ω. How many are there?

   (b) Let A be the event that the first flip is a heads. List all the outcomes in A. How many are there?

   (c) Let B be the event that the third flip is a heads. List all the outcomes in B. How many are there?

   (d) Let C be the event that the first and third flip are heads. List all outcomes in C. How many are there?

   (e) Let D be the event that the first or the third flip is heads. List all outcomes in D. How many are there?

   (f) Are the events A and B disjoint? Express C in terms of A and B. Express D in terms of A and B.

   (g) Suppose now the coin is flipped $n \geq 3$ times instead of 4 flips. Compute $|\Omega|, |A|, |B|, |C|, |D|$.

   (h) Your gambling buddy found a website online where he could buy trick coins that are heads or tails on both sides. He puts three coins into a bag: one coin that is heads on both sides, one coin that is tails on both sides, and one that is heads on one side and tails on the other side. You shake the bag, draw out a coin at random, put it on the table without looking at it, then look at the side that is showing. Suppose you notice that the side that is showing is heads. What is the probability that the other side is heads? Show your work. (Hint: the answer is NOT 1/2).

2. **To Be Fair (5 points)**

   Suppose you have a biased coin with $P(heads) \neq 0.5$. How could you use this coin to simulate a fair coin? (Hint: Think about pairs of tosses.)

3. **Picking CS Classes (2/3/5 points)**

   The EECS (Elegant Etiquette Charm School) department has d different classes being offered in Summer 2016. These include classes such as dressing etiquette, dining etiquette, and social etiquette, etc. Let's assume that all the classes are equally popular and each class has essentially unlimited seating! Suppose that c students are enrolled this semester and the registration system, EleBEARS (Elegant Bears), requires each student to choose a class s/he plans to attend.

   (a) What is the probability that a given student chooses the first class, dressing etiquette?

   (b) What is the probability that a given class is chosen by no student?

   (c) If there are $d = 20$ classes, what should c be in order for the probability to be at least one half that (at least) two students enroll in the same class?

4. **Poisoned pancakes (2/3/5 points)**

   You have been hired as an actuary by IHOP corporate headquarters, and have been handed a report from Corporate Intelligence that indicates that a covert team of ninjas hired by Denny's will sneak into some IHOP, and will have time to poison ten of the pancakes being prepared (they can't stay any longer to avoid being discovered by Pancake Security). Given that an IHOP kitchen has 100 pancakes being prepared, and there are twenty patrons, each ordering five pancakes (which are chosen uniformly at random from the pancakes in the kitchen), calculate the probabilities that a particular patron:

   (a) will not receive any poisoned pancakes;
   (b) will receive exactly one poisoned pancake;
   (c) will receive at least one poisoned pancake.

5. **Cliques in random graphs (3/5/7 points)**

   Consider a graph $G(V,E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads. So for example if $n = 2$, then with probability 1/2, $G(V,E)$ is the graph consisting of two vertices connected by an edge, and with probability 1/2 it is the graph consisting of two unconnected vertices.

   (a) What is the size of the sample space?
   (b) A k-clique in graph is a set of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example a 3-clique is a triangle. What is the probability that a particular set of k vertices forms a k-clique?
   (c) Prove that the probability that the graph contains a k-clique for $k = 4\lceil \log n \rceil + 1$ is at most $1/n$.

6. **Drunk man (1/2/2/2/3 points)**

   Imagine that you have a drunk man moving along the horizontal axis (that stretches from $x = -\infty$ to $x = +\infty$). At time $t = 0$, his position on this axis is $x = 0$. At each time point $t = 1$, $t = 2$, etc., the man moves forward (that is, $x(t+1) = x(t) + 1$) with probability 0.5, backward (that is, $x(t+1) = x(t) - 1$) with probability 0.3, and stays exactly where he is (that is, $x(t+1) = x(t)$) with probability 0.2.

   (a) What are all his possible positions at time t, $t \geq 0$?
   (b) Calculate the probability of each possible position at $t = 1$.
   (c) Calculate the probability of each possible position at $t = 2$.
   (d) Calculate the probability of each possible position at $t = 3$.
   (e) If you know the probability of each position at time t, how will you find the probabilities at time $t+1$?

   Note: The following parts are optional.
   The Drunk Man has regained some control over his movement, and no longer stays in the same spot; he only moves forwards or backwards. More formally, let the Drunk Man's initial position be $x(0) = 0$. Every second, he either moves forward one pace or backwards one pace, *i.e.*, his position at time $t+1$ will be one of $x(t+1) = x(t) + 1$ or $x(t+1) = x(t) - 1$.

   We want to compute the number of paths in which the Drunk Man returns to 0 at time t and it is his first return, *i.e.*, $x(t) = 0$ and $x(s) \neq 0$ for all s where $0 < s < t$. Note, we **no longer** care about probabilities. We are just counting paths here.

(a) How many paths can the Drunk Man take if he returns to 0 at $t = 6$ and it is his first return?

(b) How many paths can the Drunk Man take if he returns to 0 at $t = 7$ and it is his first return?

(c) How many paths can the Drunk Man take if he returns to 0 at $t = 8$ and it is his first return?

(d) How many paths can the Drunk Man take if he returns to 0 at $t = 2n+1$ for $n \in \mathbb{N}$ and it is his first return?

(e) How many paths can the Drunk Man take if he returns to 0 at $t = 2n+2$ for $n \in \mathbb{N}$ and it is his first return? (Hint: read http://en.wikipedia.org/wiki/Catalan_number and use any result there if you need.)

7. **Independence (2/4/4 points)**

   (a) **Independence (due to H.W. Lenstra)**

   Suppose we pick a random card from a standard deck of 52 playing cards. Let A represent the event that the card is an queen, B the event that the card is a spade, and C the event that a red card (a heart or a diamond) is drawn.

   i. Which two of A, B, and C are independent? Justify your answer carefully. (In other words: For each pair of events (AB, AC, and BC), state and prove whether they are independent or not.)

   ii. What if a joker is added to the deck? Justify your answer carefully.

   (b) **Independence (due to H.W. Lenstra)**

   Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events. Let $\bar{A} = \Omega - A$ and $\bar{B} = \Omega - B$ (sometimes written $\neg A$ and $\neg B$) denote the complementary events. For the purposes of this question, you may use the following definition of independence: Two events A, B are *independent* if $Pr[A \cap B] = Pr[A]Pr[B]$.

   i. Prove or disprove: $\bar{A}$ and $\bar{B}$ are necessarily independent.

   ii. Prove or disprove: A and $\bar{B}$ are necessarily independent.

   iii. Prove or disprove: A and $\bar{A}$ are necessarily independent.

   iv. Prove or disprove: It is possible that $A = B$.

   (c) **Bonferroni's inequalities**

   i. For events A, B in the same probability space, prove that

   $$Pr[A \cap B] \geq Pr[A] + Pr[B] - 1.$$

   ii. Generalize part (a) to prove that, for events $A_1, \ldots, A_n$ in the same probability space (and any n),

   $$Pr[A_1 \cap \cdots \cap A_n] \geq Pr[A_1] + \cdots + Pr[A_n] - (n-1).$$

8. **Birthdays (2/3/5 points)**

   Suppose you record the birthdays of a large group of people, one at a time until you have found a match, i.e., a birthday that has already been recorded. (Assume there are 365 days in a year.)

   (a) What is the probability that it takes more than 20 people for this to occur?

   (b) What is the probability that it takes exactly 20 people for this to occur?

(c) Suppose instead that you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday matches your own birthday. What is the probability that it takes exactly 20 people for this to occur?

9. **Blood Type (2/3/5 points)**
Consider the three alleles, A, B, and O, for human blood types. As each person inherits one of the 3 alleles from each parent, there are 6 possible genotypes: AA, AB, AO, BB, BO, and OO. Blood groups A and B are dominant to O. Therefore, people with AA or AO have type A blood. Similarly, BB and BO result in type B blood. The AB genotype is called type AB blood, and the OO genotype is called type O blood. Each parent contributes one allele randomly. Now, suppose that the frequencies of the A, B, and O alleles are 0.4, 0.25, and 0.35, respectively, in Berkeley. Alice and Bob, two residents of Berkeley are married and have a daughter, Mary. Alice has blood type AB.

   (a) What is the probability that Bob's genotype is AO?

   (b) Assume that Bob's genotype is AO. What is the probability that Mary's blood type is AB?

   (c) Assume Mary's blood type is AB. What is the probability that Bob's genotype is AA?

10. **Expressions (2/4/1/2/1 points)**

For each problem, just write down a mathematical expression.There is no need to justify/explain/derive the answer.

   (a) **Bayes Rule - Man Speaks Truth**

      i. A man speaks the truth 3 out of 4 times. He flips a biased coin that comes up Heads $\frac{1}{3}$ the time and reports it's Heads. What is the probability it is Heads?

      ii. A man speaks the truth 3 out of 4 times. He rolls a fair 6-sided dice and reports it comes up 6. What is the probability it is really 6?

   (b) **Unlikely events**

      i. Toss a fair coin x times. What is the probability that you never get heads?

      ii. Roll a fair die x times. What is the probability that you never roll a six?

      iii. Suppose your weekly local lottery has a winning chance of $1/10^6$. You buy lottery from them for x weeks in a row. What is the probability that you never win?

      iv. How large must x be so that you get a head with probability at least 0.9? Roll a 6 with probability at least 0.9? Win the lottery with probability at least 0.9?

   (c) **Roll Dice**
   You roll three fair six-sided dice. What is the probability of rolling a triple (all three dice agree)? What is the probability of rolling a double (two of the dice agree with each other)?

   (d) **Lie Detector**
   A lie detector is known to be 80% reliable when the person is guilty and 95% reliable when the person is innocent. If a suspect is chosen from a group of suspects of which only 1% have ever committed a crime, and the test indicates that the person is guilty, what is the probability that he is innocent?

   (e) **Chess Squares**
   Two squares are chosen at random on 8×8 chessboard. What is the probability that they share a side?