CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Dinh, Psomas, and Ye HW 3

Due Monday July 11 at 1:59PM

1. (12 points: 3/3/6) **Planar Graph**

   A simple graph is *triangle-free* when it has no simple cycle of length three.

   (a) Prove for any connected triangle-free planar graph with $v > 2$ vertices and e edges, $e \leq 2v - 4$. *Hint*: Similar to the proof that $e \leq 3v - 6$.

   (b) Show that any connected triangle-free planar graph has at least one vertex of degree three or less.

   (c) Prove by induction on the number of vertices that any connected triangle-free planar graph is 4-colorable. *Hint*: use part b

2. (16 points: 5/4/4/3) **Countability**

   We say that a set S is **countable** if there is a bijection from $\mathbb{N}$, or a subset of $\mathbb{N}$, to S. Equivalently, S has the same cardinality as $\mathbb{N}$ or a subset of $\mathbb{N}$.

   For the following, you may find these facts useful (you should be able to prove them from the definition above, **but you don't need to**).

   If there exists a surjection from $\mathbb{N}$ to S, then S is countable. If there exists an injection from S to $\mathbb{N}$, then S is countable.

   (a) Given two countable sets A and B, prove that the Cartesian Product $A \times B = \{(a,b) : a \in A, b \in B\}$ is countable.

   (Hint: This should be reminiscent of the argument to show why rational numbers are countable.)

   (b) Consider $\mathbb{Z}^m$, the set of all m-length vectors with integer elements for some positive integer m. Prove that $\mathbb{Z}^m$ is countable.

   (Hint: Use induction.)

   Quick note: Length m vectors here means a vector with m elements.

   (c) Prove that the countable union of countable sets is countable. I.e, prove the following is countable:

   $$A_1 \cup A_2 \cup A_3 \ldots$$

   where each A_i is countable and there are countably many of them.

   (d) If there are countably infinite (also known as denumerable) sets A_i, could the above union still be finite? If yes, explain briefly what kind of sets A_i for which this holds. If not, prove that union is always infinite.

   (Hint: How does this relate to Cartesian Products?)

3. (20 points:1/1/1/1/2/2/2/2/2/2/2/2) **Countability**

   For each of the following sets, determine and briefly explain whether it is finite, countably infinite (i.e., countable and infinite), or uncountably infinite (i.e., uncountable)

   (a) $\mathbb{R}$ (the set of all real numbers)

   (b) $\mathbb{C}$ (the set of all complex numbers)

   (c) $\{0,1,2\}^*$ (the set of all finite-length ternary strings)

   (d) $\mathbb{Z}^3 = \{(a,b,c) : a,b,c \in \mathbb{Z}\}$ (the set of triples of integers)

   (e) $S = \{p(x) : p(x) = ax^2 + bx + c, \text{ where } a,b,c \in \mathbb{Z}\}$ (the set of all polynomials of degree at most 2 with integer coefficients)

   (f) $T = \{p(x) : p(x) = a_n x^n + \cdots + a_1 x + a_0, \text{ where } n \in \mathbb{N} \text{ and } a_0, a_1, \ldots, a_n \in \mathbb{Z}\}$ (the set of all polynomials with integer coefficients, of any degree)

   (g) Numbers that are the roots of nonzero polynomials with integer coefficients.

   (h) $U = \{f : \mathbb{N} \to \{0,1\}\}$ (the set of all functions that map each natural number to 0 or 1)

   (i) $V = \{f : \mathbb{N} \to \mathbb{N}\}$ (the set of all functions that map each natural number to a natural number)

   (j) Computer programs that halt.

   (k) Computer programs that always correctly tell if a program halts or not.

   (l) Computer programs that correctly return the product of their two integer arguments.

4. (21 points: 1/1/1/1/1/1/1/1/1/1/1/2/2/2/2/2) **Counting, counting, and more counting**

   The only way to learn counting is to practice, practice, practice, so here is your chance to do so. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

   (a) How many 10-bit strings are there that contain exactly 4 ones?

   (b) How many different 13-card bridge hands are there? (A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.)

   (c) How many different 13-card bridge hands are there that contain no aces?

   (d) How many different 13-card bridge hands are there that contain all four aces?

   (e) How many different 13-card bridge hands are there that contain exactly 6 spades?

   (f) How many 99-bit strings are there that contain more ones than zeros?

   (g) How many different anagrams of FLORIDA are there? (An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.)

   (h) How many different anagrams of ALASKA are there?

   (i) How many different anagrams of ALABAMA are there?

   (j) How many different anagrams of MONTANA are there?

   (k) If we have a standard 52-card deck, how many ways are there to order these 52 cards?

   (l) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(m) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(n) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

(o) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(p) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student?

5. (11 points: 4/7) **Charm School Applications**

(a) n males and n females apply to the Elegant Etiquette Charm School (EECS) within UC Berkeley. The EECS department only has n seats available. In how many ways can it admit students? Use the above story for a combinatorial argument to prove the following identity:

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

(b) Among the n admitted students, there is at least one male and at least one female. On the first day, the admitted students decide to carpool to school. The boy(s) get in one car, and the girl(s) get in another car. Use the above story for a combinatorial argument to prove the following identity:

$$\sum_{k=1}^{n-1} k \cdot (n-k) \cdot \binom{n}{k}^2 = n^2 \cdot \binom{2n-2}{n-2}$$

(Hint: Each car has a driver...)

6. (13 points: 2/3/4/4) **Getting to CS**

Harry, the chosen one, is chosen to drive the boys to the Charm School. The Flying Ford Anglia he's driving is behaving weirdly – it would only go south or east for at least a certain distance. Figure 1 shows the path the car could go from Harry's house (H) to the Charm School (CS).

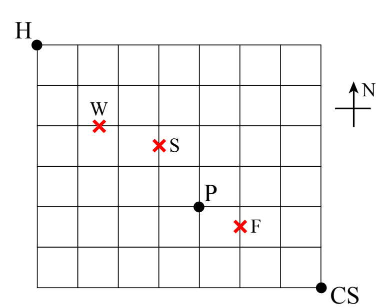

Figure 1: The map from Harry's house to the Charm School

(a) How many ways can he get there?

(b) Harry has to pick up other students at point P. How many ways can he stop by point P and go to the Charm School?

(c) The Whomping Willow (W) will attack anything that comes near. Harry must not drive through it. How many ways can he pick up the students and then go to the Charm School now?

(d) On top of the Whomping Willow, the Marauder's Map shows Professor Snape (S) and Filch (F) whom he doesn't want to drive past either. How many ways can Harry go to the Charm School without getting past the Whomping Willow (W), Professor Snape (S), or Filch (F), while still picking up other students?

7. (7 points: 3/4) **Charming Star**

   At the end of each day, students will vote for the most charming student. There are 5 candidates and 100 voters. Each voter can only vote once, and all of their votes weigh the same.

   (a) How many possible voting combinations are there for the 5 candidates?

   (b) How many possible voting combinations are there such that exactly one candidate gets more than 50 votes?