CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Dinh, Psomas, and Ye Midterm 2

PRINT Your Name: ______________________, ______________________
(last) (first)

By signing below, I agree that (a) I will not give or receive help from others during the exam, (b) that I will not use any electronic devices or other prohibited resources during the exam, and (c) that I will not discuss or disclose the contents of the exam before 1:30 PM PDT on Friday, July 29, 2016.

SIGN Your Name: ______________________ DATE: ______________

PRINT Your Student ID: ______________________

CIRCLE your exam room: 1 Pimentel 120 Latimer 606 Soda 380 Soda Other

Name of the person sitting to your left: ______________________

Name of the person sitting to your right: ______________________

- After the exam starts, please write your student ID (or name) on every sheet (we will remove the staple when scanning your exam).
- We will not grade anything outside of the space provided for a problem unless we are clearly told in the space provided for the question to look elsewhere.
- You may consult one **handwritten** double-sided (or two single-sided) sheet of notes. Apart from that, you may not look at books, notes, etc. Calculators, phones, computers, and other electronic devices are not permitted.
- There are 18 pages (9 sheets) on the exam, including 2 pages (1 sheet) of extra paper and an 2-page (1 sheet) equation reference at the end containing tables of distributions and bounds. Notify a proctor immediately if a page is missing.
- There are eight questions on this exam, worth a total of 190 points.
- **You may, without proof, use theorems and facts that were proven in the notes and/or in lecture.**
- **You have 110 minutes.**

Do not turn this page until your proctor tells you to do so.

1 T/F

(4 points each) Circle T for True or F for False. We will only grade the answers, and are unlikely to even look at any justifications or explanations.

(a) T F Given some sample space $\Omega = \{1,2,3\}$, and events $A = \{1,2\}$ and $B = \{1\}$, then $Pr[B|A] = \frac{1}{2}$.

(b) T F Given some sample space $\Omega = \{1,2,3\}$, and events $A = \{1,2\}$ and $B = \{1\}$, then $Pr[B|A] = \frac{1}{3}$.

(c) T F Linearity of expectation applies if and only if the random variables involved are independent.

(d) T F The variance of a random variable that only attains values in the interval $[-1,1]$ is at most 1.

(e) T F If two events are disjoint, they are independent.

(f) T F If two events are independent, they are disjoint.

(g) T F The value v which maximizes the probability density function (pdf) of a continuous random variable X is equal to $E[X]$.

(h) T F A Markov Chain always has a stationary distribution.

2 To Catch a Magikarp

If your solutions involve computing integrals, **you do not have to do the calculations**.

Supose that when you catch a pokemon, it has probability p of being a Magikarp. (A Magikarp is a type of pokemon.) You may assume that different people catch pokemon independently.

(a) (3 pts) What is the expected number of pokemon you have to catch before you catch a Magikarp?

(b) (7 pts) Suppose you and your friend are out catching pokemon. Each of you catches one pokemon per minute. Both of you continue to catch pokemon until both of you have found Magikarps (so the person who catches a Magikarp first will continue to catch pokemon until the other person has also found a Magikarp). What is the expected time it takes for you to stop? Express your answer in closed form (i.e. not an infinite sum). You may use the fact that $\sum_{i=0}^{\infty} c^k = 1/(1-c)$ for $0 < c < 1$.

(c) (3 pts) Suppose that you catch a pokemon with probability q every minute (you never catch more than one pokemon in a minute). What is the distribution (including parameters) of the time before you catch your first Pokemon?

(d) (3 pts) Suppose again that you catch a pokemon with probability q every minute (you never catch more than one pokemon in a minute). What is the distribution (including parameters) of the time before you catch your first Magikarp?

(e) (3 pts) Suppose that people catch pokemon until they catch a Magikarp, at which point they stop. What is the expected number of pokemon you and your $100,000$ friends have to catch (in total) before you all catch Magikarp?

(f) (7 pts) Suppose again that people catch pokemon until they catch a Magikarp, at which point they stop. What is the probability that the total number of pokemon caught by you and your $100,000$ friends is greater than x? Compute the best bound you can, assuming the Central Limit Theorem applies.

3 Good Proof, Bad Proof

For each of the following propositions and proofs, indicate which of the following cases apply:

1. Correct proposition with correct proof. No further explanation is needed for this case.
2. Correct proposition but incorrect proof. In this case, identify what the error in the proof is and provide a correct proof.
3. Incorrect proposition (therefore the proof is clearly incorrect). In this case, identify what the error in the proof is and provide a counterexample to the proposition.

(a) (10 points) **Let X be a random variable with expectation 1 and variance 1. Then $Pr[X \geq 7] \leq \frac{1}{36}$**

Proof. $Pr[X \geq 7] = Pr[X - 1 \geq 6] = Pr[|X-1| \geq 6] = Pr[|X - E(X)| \geq 6] \leq \frac{Var(X)}{6^2} = \frac{1}{36}$ □

(b) (10 points) **Suppose i is a state in a finite, irreducible, aperiodic Markov chain with transition matrix P and states $\{0, ..., n\}$. Suppose that at timestep 1, the Markov chain is distributed according to its stationary distribution π. Then in the next timestep, 2, the probability that we leave i (i.e. that we started in i at timestep 1, and go to something that's not i at timestep 2), is the same as the probability that we enter i (i.e. we started outside i at timestep 1 and are at i at timestep 2).**

Proof. The probability that we start at i is just π_i. The probability that we leave i if we started at i is $\sum_{j \neq i} P_{i,j}$. Therefore, the probability that we leave i is $\pi_i \sum_{j \neq i} P_{i,j}$.

The probability that we start at some state j (that's not i) is π_j, and the probability that we go from j to i during the timestep is $P_{j,i}$. Therefore, the probability that we enter i is $\sum_{j \neq i} \pi_j P_{j,i}$.

Since π is a stationary distribution, $\pi_i = \sum_{j=0}^{n} \pi_j P_{j,i}$. We also know that, by definition of a Markov chain, $\sum_{j=0}^{n} P_{i,j} = 1$. Therefore, $\pi_i = \pi_i * 1 = \pi_i \sum_{j=0}^{n} P_{i,j}$.

Therefore, $\pi_i \sum_{j=0}^{n} P_{i,j} = \sum_{j=0}^{n} \pi_j P_{j,i}$. Subtracting $\pi_i P_{i,i}$ from both sides gives us $\sum_{j \neq i} \pi_j P_{j,i} = \sum_{j \neq i} \pi_i P_{i,j}$, so the probability that we leave i and the probability that we enter i are the same. □

4 Markov Chains

Suppose you play the following game: You toss a fair six-sided die repeatedly until the same number comes up twice in a row, whereupon you stop.

(a) (4 points) Draw a Markov chain corresponding to this process with three states: a single state corresponding to the start of the process (before we've tossed any dice), a single state corresponding to the case when we've finished, and one more state.

(b) (4 points) Is this Markov chain aperiodic? Why or why not?

(c) (4 points) Is this Markov chain irreducible? Why or why not?

(d) (4 points) Does this Markov chain have a stationary distribution? If it does, tell us what it is. If no stationary exists, why not?

(e) (4 points) Write down the transition matrix for this Markov chain.

(f) (5 points) How many tosses should we expect to do before we stop?

5 Money bags

I have a bag containing either a \$1 or a \$5 bill (with equal probability assigned to both possibilities).

(a) (4 points) How much money would you be willing to pay for this bag? In other words, what amount of money and this bag would you be indifferent between? This amount should make your expected profit zero, and your expected loss zero.

(b) (4 points) I add a \$1 bill to the bag, so it now contains two bills. The bag is shaken. I draw out a random bill out of the bag, and it is a \$1 bill. How much money are you willing to pay now for this bag?

(c) (5 points) Consider the original bag (before part (b)). Your friend, who lies with probability 0.7, takes a look inside the bag, and tells you that it has the \$5 bill. How much money are you willing to pay now?

(d) (7 points) A company is selling these bags (same as the original one - each bag contains \$1 or \$5 with equal probability, and the amount of money in each bag is mutually independent of the amount in the others) for \$2. You decide to buy $1,000,000$ of these bags, hoping to make a profit. Find an upper bound on the probability that you lose money. Find a linear bound for 2 points, a quadratic bound for 4 points, or an exponential bound for full credit.

6 Squared Exponential

Suppose x is distributed exponentially with parameter λ.

(a) (5 points) What is the probability that x is in the interval $[t, t+\varepsilon]$ for infinitesimally small ε? Express your answer in terms of t and ε.

(b) (8 points) If $x^2 \in [q, q+\delta]$ for infinitesimally small δ, what interval must x fall into? Express your answer in terms of q and δ. *Hint: If you are dealing with an infinitesimally small number s and encounter a polynomial in s, you can drop all but the constant and first-order terms. In other words, if s is infinitesimally small, you can approximate* $a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots$ *as* $a_0 + a_1 s$.

(c) (7 points) What is the pdf of the distribution of x^2?

7 Bounds

(a) (7 pts) Show by example that Markov's inequality is tight; that is, show that given $k > 0$, there exists a discrete nonnegative random variable X such that $\Pr[X \geq k] = \mathbf{E}[X]/k$.

(b) (7 pts) Show by example that Chebyshev's inequality is tight; that is, show that given $k > 0$, here exists a random variable X such that $\Pr[|X - \mathbf{E}[X]| \geq k\sigma(X)] = 1/k^2$.

(c) (7 pts) Show that there is no random variable X, that takes values in some finite set $\{v_1, \ldots, v_N\}$, such that for all $k > 0$, Markov's inequality is tight; that is, $\Pr[X \geq k] = \mathbf{E}[X]/k$.

8 Boolean dot product

Let $C = (A_1 \wedge B_1) \vee (A_2 \wedge B_2) \vee \cdots \vee (A_N \wedge B_N)$. A_i's and B_i's are i.i.d Bernoulli r.v. with parameter $\frac{1}{2}$. **Express your answers as an algebraic expression of numbers and N.**

(a) (13 points) What is the probability that A_1 is *true* provided that C is *true*?

(b) (13 points) Suppose we pick a variable uniformly at random from the set $\{A_1, \ldots A_N, B_1, \ldots, B_N\}$ and set it to *false*. The remaining variables are i.i.d. Bernoulli with parameter $\frac{1}{2}$. What is the probability that A_1 is *true*, provided that C is true?

9 Extra Pages

If you use this page as extra space for answers to problems, please indicate clearly which problem(s) you are answering here, and indicate **in the original space for the problem** that you are continuing your work on an extra sheet. You can also use this page to give us feedback or suggestions, report cheating or other suspicious activity, or to draw doodles.

More extra paper. If you fill this sheet up you can request extra sheets from a proctor (just make sure to write your SID on each one, and to staple the extra sheets to your exam when you submit it).

Reference Sheet for Distributions and Bounds

Discrete Distributions

Bernoulli Distribution

- 1 with probability p, 0 with probability $1-p$
- Expectation: p
- Variance: $p(1-p)$

Binomial Distribution with parameters n, p

- $\Pr[X=k] = \binom{n}{k} p^k (1-p)^{n-k}$
- Expectation: np
- Variance: $np(1-p)$

Geometric Distribution with parameters p

- $\Pr[X=k] = (1-p)^{k-1} p$
- Expectation: $1/p$
- Variance: $\frac{1-p}{p^2}$

Uniform Distribution with parameters a, b ($a \le b$)

- $\Pr[X=k] = \frac{1}{b-a+1}$ for $k \in [a,b]$, 0 otherwise.
- Expectation: $(a+b)/2$
- Variance: $\frac{(b-a+1)^2-1}{12}$

Poisson Distribution with parameter λ

- $\Pr[X=k] = \frac{\lambda^k e^{-\lambda}}{k!}$
- Expectation: λ
- Variance: λ

Continuous Distributions

Uniform Distribution with parameters a, b ($a < b$).

- PDF: $\frac{1}{b-a}$ for $x \in [a,b]$, 0 otherwise.
- Expectation: $(a+b)/2$

- Variance: $\frac{(b-a)^2}{12}$

Exponential Distribution with parameter λ

- PDF: $\lambda e^{-\lambda x}$ for $x > 0$, 0 otherwise
- Expectation: $1/\lambda$
- Variance: $1/\lambda^2$

Normal Distribution with parameters μ, σ^2

- PDF: $\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
- Expectation: μ
- Variance: σ^2

Chernoff Bounds

Theorem: Let $X_1, \ldots, X_n$ be independent indicator random variables such that $Pr[X_i = 1] = p_i$, and $Pr[X_i = 0] = 1 - p_i$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$. Then the following Chernoff bounds hold:

- For any $\delta > 0$:
$$Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$
- For any $1 > \delta > 0$:
$$Pr[X \leq (1-\delta)\mu] \leq \left(\frac{e^{\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu}$$
- For any $1 > \delta > 0$:
$$Pr[X \geq (1+\delta)\mu] \leq e^{-\frac{\mu\delta^2}{3}}$$
- For any $1 > \delta > 0$:
$$Pr[X \leq (1-\delta)\mu] \leq e^{-\frac{\mu\delta^2}{2}}$$
- For $R > 6\mu$:
$$Pr[X \geq R] \leq 2^{-R}$$