CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Dinh, Psomas, and Ye Midterm 1

PRINT Your Name: ________________________________, ________________________________
(last) (first)

By signing below, I agree that (a) I will not give or receive help from others during the exam, (b) that I will not use any electronic devices or other prohibited resources during the exam, and (c) that I will not discuss or disclose the contents of the exam before 1:30 PM PDT on Friday, July 8, 2016.

SIGN Your Name: ______________________________________ DATE: ______________________

PRINT Your Student ID: ______________________________________

CIRCLE your exam room: 1 Pimentel 120 Latimer 606 Soda 380 Soda Other

Name of the person sitting to your left: ______________________________________

Name of the person sitting to your right: ______________________________________

- After the exam starts, please write your student ID (or name) on every odd page (we will remove the staple when scanning your exam).
- We will not grade anything outside of the space provided for a problem unless we are clearly told in the space provided for the question to look elsewhere.
- You may consult one handwritten single-sided sheet of notes. Apart from that, you may not look at books, notes, etc. Calculators, phones, computers, and other electronic devices are prohibited.
- There are 16 pages (8 sheets) on the exam, including 2 pages (1 sheet) of extra paper. Notify a proctor immediately if a page is missing.
- There are eight questions on this exam, worth a total of 190 points.
- **You may, without proof, use theorems and facts that were proven in the notes and/or in lecture.**
- **You have 110 minutes.**

Do not turn this page until your proctor tells you to do so.

1 T/F

(3 points each) Circle T for True or F for False. We will only grade the answers, and are unlikely to even look at any justifications or explanations.

(a) T F $(P \to (Q \vee R)) \equiv \overline{(P \wedge \overline{Q}) \wedge (P \wedge \overline{R})}$.

(b) T F $\exists x \forall y P(x,y) \to \exists y \exists x P(x,y)$

(c) T F There exists some n such that for all $k > n$, the complete graph with k vertices is not planar.

(d) T F The traditional marriage algorithm (males propose) never produces a female-optimal matching.

(e) T F Suppose we have n men and n women, for $n > 2$. Then there exist some preference lists that causes the traditional marriage algorithm to halt in exactly two steps (i.e. everyone is matched by the end of the second day).

(f) T F If an undirected graph does not contain cycles, it's a tree.

(g) T F Complete undirected graphs cannot be trees.

(h) T F It is impossible to develop a general test that determines whether or not a program returns the string "CS70".

(i) T F Suppose you are given program P that takes a single input. No matter what P is, it is impossible to come up with a program M that tells you whether or not P halts on input x.

2 Counting

Clearly indicate your correctly formatted answer: this is what is to be graded. You may leave simple mathematical expressions, including binomial coefficients and factorials, un-evaluated. We will only grade the answers, and are unlikely to even look at any justifications or explanations.

(a) **(5 points)** Suppose $x_1 + x_2 + x_3 + .. + x_{10} = 20$ where x_i $(1 \leq i \leq 10)$ are all natural numbers (that is, non-negative integers). How many distinct solutions are there for this equation?

(b) **(5 points)** In a party, we have n men and n women. In how many different ways can we pair them up?

(c) **(5 points)** We have 3 TAs covering a 3-hour homework party, which is divided into three slots, each one hour long. Each TA can sign up for anywhere from 0 to 3 slots, as long as there is **at least one TA assigned to each slot**. In how many ways can they sign up for this homework party?

3 Proofs

(a) **(7 points)** Show that there exists some numbers $k, m \in \mathbb{R}$ such that for all numbers $n \in \mathbb{R}$: $nk = m$

(b) **(7 points)** Show that during a party where people shake hands with some other guests, the number of guests who shake hands with odd number of guests is even.

(c) **(7 points)** Suppose you are proving a proposition $P(n)$ by induction on n. You successfully prove the induction step, $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$. But then you notice that $P(2501)$ is false. Can you conclude anything about $P(25)$? Justify your answer.

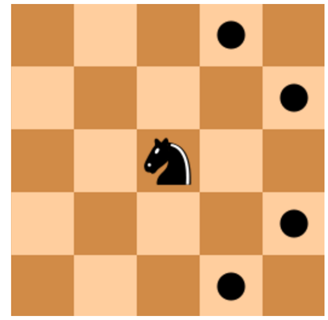
(d) **(7 points)** For all $n \geq k \geq 0$, $(n-k)\binom{n}{k} = n\binom{n-1}{k}$ *(hint: combinatorial proof)*.

4 Knight Rider

Suppose we have an infinite grid. A knight starts at coordinate $(0,0)$, and each turn, can move ± 2 steps horizontally and ± 1 steps vertically, or ± 1 steps horizontally and ± 2 steps vertically. For instance, in the following diagram, the knight at the center can move into any grid point marked with a black circle.

(a) **(12 points)** Prove that the knight can reach every square (a,b) in the grid, in under $3(a+b)$ steps for **full credit** (12 points). Prove that the knight can reach every square in the grid for **partial credit** (7 points). It might be easier to prove the statement by induction, but all proofs will be accepted.

(b) Now suppose our knight is crippled and can only move to the right. This means that there are only 4 possible moves: 2 steps right and ± 1 step vertically, or ± 2 steps vertically and 1 step right, as shown in the diagram to the right. Consider the (infinite) *directed* graph where vertices are points the knight can reach and edge (u,v) means that the knight can move from u to v in one move. Answer the following questions (with justification):



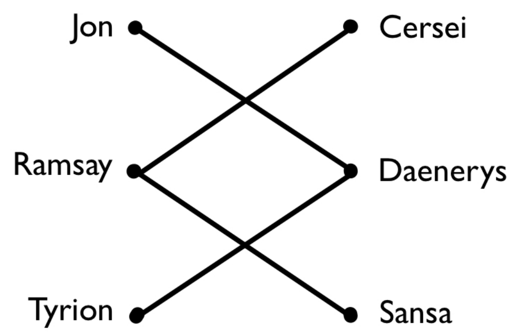
(i) **(6 points)** Is the graph acyclic?

(ii) **(6 points)** For every vertex v, is there a unique path from the origin to v?

5 Stable Matching

In this problem, we are given a bipartite graph: $G = (M, W, E)$ where there are two sets of vertices, M and W, and $E \subseteq M \times W$; that is, each edge is incident to a vertex in M and a vertex in W. We also know that $|M| = |W| = k$. The graph corresponds to an input to the stable marriage problem, with k men (represented by vertices in M) and k women (represented by vertices in W). Every man $v \in M$ prefers every woman in the set S_v that v is adjacent to in G over all women corresponding to vertices in $W \setminus S_v$. Likewise, every woman prefers any man she is adjacent to in G over any man she is not adjacent to.

A *perfect matching* is a set of edges where every vertex is incident to exactly one edge in the matching. Another view is that each vertex is matched to another vertex; similar to a pairing in stable marriage except that the pair must correspond to an edge in the graph. We say that a graph has a perfect matching if a subset of its edges forms a perfect matching.



In the above graph, Cersei prefers Ramsay (whom she is adjacent to) over both Jon and Tyrion (who do not have edges connecting them with Cersei). Note that this graph does not say anything about whether Jon or Tyrion ranks higher in Cersei's preference list, only that Ramsay is higher than both of them. Also, note that this graph has no perfect matching.

(a) **(12 points)** Prove that if G has a perfect matching, then there exists a preference list such that the traditional marriage algorithm results in a perfect matching.

(b) **(12 points)** Disprove by counterexample: If G has a perfect matching, then for all preference lists consistent with G, the traditional marriage algorithm results in a perfect matching. That is, find a graph G, with a perfect matching, and a preference list consistent with G (every person prefers people that he/she is adjacent to in the graph over people that he/she is not adjacent to), such that if we run the traditional marriage algorithm on the preference list, the output matching of men and women will not be a perfect matching in G.

6 Good Proof, Bad Proof

For each of the following propositions and proofs, indicate which of the following cases apply:

1. Correct proposition with correct proof. No further explanation is needed for this case.
2. Correct proposition but incorrect proof. In this case, identify what the error in the proof is and provide a correct proof.
3. Incorrect proposition (therefore the proof is clearly incorrect). In this case, identify what the error in the proof is and provide a counterexample to the proposition.

(a) **(9 points) If $n \geq 2$, n can be expressed as the sum of distinct prime numbers.**

Proof. By induction on n. For the base case, $n = 2$, the proposition immediately follows from the fact that 2 is prime. Now suppose that the statement is true for all $2 < n' < n$, that is n' can be written as the sum of distinct prime numbers. Therefore $n - 2$ can be written as the sum of distinct primes. So we can write n as the sum of 2 and the distinct primes that sum up to $n - 2$ (which we know exist by the inductive hypothesis); therefore, the proposition also holds for n. □

(b) **(9 points) A graph G with n vertices and $n-1$ edges contains no cycles.**

Proof. Let G be a graph with n vertices and $n-1$ edges. We show that G contains no cycles. Assume to the contrary that G contains cycles. Remove an edge from a cycle. This leaves a connected graph on n vertices with $n-2$ edges which is impossible as a connected graph on n vertices must at least have $n-1$ edges. □

(c) **(9 points) $\sum_{i=1}^{n} \frac{1}{i}$ goes to infinity as n grows**

Proof. $\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \leq 1 + 1 + 1 \cdots = n$, which goes to infinity as n goes to infinity. □

7 Countability

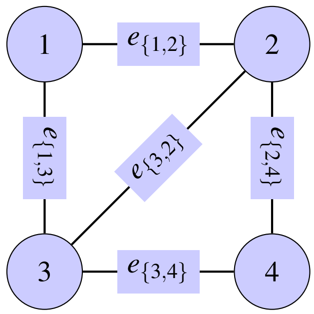
Which of the following are countable, and which are uncountable? Justify your answer.

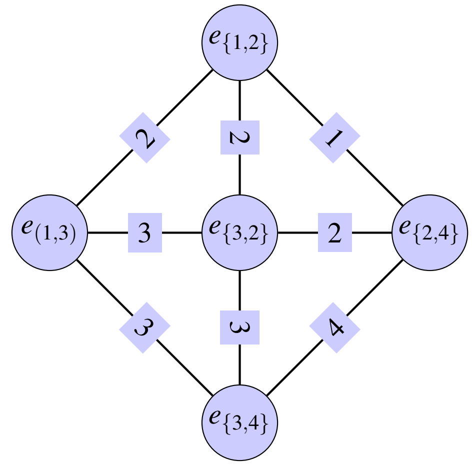
(a) **(7 points)** The set of all graphs.

(b) **(7 points)** The set of all subsets of $\mathbb{N} \setminus \{0\}$.

(c) **(7 points)** The set of all programs that halt.

8 Edge complement





The **edge complement** graph of a graph $G = (V,E)$ is a graph $G' = (V',E')$, such that $V' = E$, and $(i, j) \in E'$ if and only if i and j had a common vertex in G. In the above picture, the graph on the right is the edge complement of the graph on the left: for every edge $e_{\{i,j\}}$ in the graph on the left there is a vertex in the graph on the right. If two edges $e_{\{i,j\}}$ and $e_{\{j,k\}}$ share a vertex j on the left, then the corresponding vertices on the right have an edge j connecting them.

(a) **(12 points)** Prove or disprove: if a graph G has an Eulerian tour, then its **edge complement** graph has an Eulerian tour.

(b) **(12 points)** Prove or disprove: if a graph's **edge complement** graph G' has an Eulerian tour, then graph G has an Eulerian tour.

9 Empty page for extra work, doodles, etc.

If you use this page as extra space for answers to problems, please indicate clearly which problem(s) you are answering here, and indicate **in the original space for the problem** that you are continuing your work on an extra sheet. You can also use this page to give us feedback or suggestions, report cheating or other suspicious activity, or to draw doodles.

More extra paper. If you fill this sheet up you can request extra sheets from a proctor (just make sure to write your SID on each one, and to staple the extra sheets to your exam when you submit it).