1. Fun with Binary.

Prove the following statement:

$$\forall n \in \mathbb{N}, \sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

2. Power Inequality

Use induction to prove that for all integers $n \geq 1$, $2^n + 3^n \leq 5^n$.

3. Triangle Inequality

Recall the triangle inequality, which states that for real numbers x_1 and x_2,

$$|x_1 + x_2| \leq |x_1| + |x_2|.$$

Use induction to prove the generalized triangle inequality:

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

4. False Proof

What goes wrong in the following "proof"?

Theorem: If n is an even number and $n \geq 2$, then n is a power of two.

Proof:
By induction on the natural number n. Let the induction hypothesis $IH(k)$ be the assertion that "if k is an even number and $k \geq 2$, then $k = 2^i$, where i is a natural number".

Base case: $IH(2)$ states that 2 is a power of two, which it is ($2 = 2^1$).

Inductive step: Assume that k is a number greater than 2, and that $IH(j)$ holds for all $2 \leq j < k$.

Case 1: k is odd, and there is nothing to show.

Case 2: k is even, so $k \geq 4$. Since $k \geq 4$ is an even number, $k = 2l$, with $2 \leq l < k$. Therefore we can use the induction hypothesis $IH(l)$, which asserts that $l = 2^i$ for some integer i. Thus we have $k = 2l = 2^{i+1}$, so k is a power of two. $IH(k)$ holds.