1. Set Operations

- $\mathbb{R}$, the set of real numbers.
- $\mathbb{Q}$, the set of rational numbers: $\{\frac{a}{b} : a, b \in \mathbb{Z} \wedge b \neq 0\}$.
- $\mathbb{Z}$, the set of integers. $\{\dots, -2, -1, 0, 1, 2, \dots\}$.
- $\mathbb{N}$, the set of natural numbers: $\{0, 1, 2, 3, \dots\}$.

(a) $\mathbb{N} \cap \mathbb{Z} =$

(b) $\mathbb{R} \cap \mathbb{Z} =$

(c) If $S \subseteq T$, what is $S \cap T$?

(e) $\mathbb{N} \cup \mathbb{Q} =$

(f) $\mathbb{R} \cup \mathbb{R} =$

(g) If $S \subseteq T$, what is $S \cup T$?

(h) $\mathbb{R} \setminus \mathbb{Q} =$

(i) $\mathbb{Z} \setminus \mathbb{Q} =$

(j) If $S \subseteq T$, what is $S \setminus T$?

2. Sums and Products

(a) Evaluate: $\sum_{i=0}^{3} i^2 =$

(b) Evaluate: $\prod_{i=-1}^{2} (2^i) =$

(c) True or false? $\sum_{i=0}^{2} \prod_{j=-1}^{1} (ij) = \prod_{j=-1}^{1} \sum_{i=0}^{2} (ij)$

(d) True or false? True or false? $\sum_{i=0}^{2} \prod_{j=-1}^{1} (i+j) = \prod_{j=-1}^{1} \sum_{i=0}^{2} (i+j)$

3. Boolean Logic

(a) Use truth tables to prove that $A \to B \equiv \neg(A \wedge \neg B)$.

(b) Use truth tables to show that $\neg(A \vee B) \equiv \neg A \wedge \neg B$ and $\neg(A \wedge B) \equiv \neg A \vee \neg B$. These two equivalences are known as De Morgan's Law.

4. Propositional Logic

(a) Write the following proposition in logical notation: Every nonzero rational number has a corresponding rational number such that the product of the two numbers is 1.

(b) There are no integer solutions to the equation $x^2 - y^2 = 10$.

5. Challenge Question: Boolean Matrix Multiply

Define an $m \times n$ *boolean matrix* as a grid of mn boolean variables (m rows, n columns). The variable at the ith row and jth column of boolean matrix A is denoted A_{ij}.

Define the *product* of an $m \times n$ boolean matrix A and and the $n \times p$ boolean matrix B to be the $m \times p$ boolean matrix C whose i, jth element is $C_{ij} = \bigvee_{k=0}^{n} \left(A_{ik} \wedge B_{kj}\right)$ (this notation means taking the boolean OR over a bunch of Boolean terms for $k = 0$ through n, similar to the sum and product notations).

For instance, suppose we define the matrix A as:

$$A = \begin{pmatrix} true & false \\ false & true \\ true & false \end{pmatrix}$$

and the matrix B as:

$$B = \begin{pmatrix} false & false \\ false & true \end{pmatrix}.$$

a) What is AB?

b) If I tell you that the ith row of A is entirely comprised of $false$ values, what can you tell me about the ith row of AB?

c) What if I tell you that the ith row of A is entirely composed of $true$ values?

We can also define matrices over real numbers similarly (where the values in the grid are real numbers instead of booleans), with products of $W = XY$ of matrices X, Y over real numbers being defined as $W_{ij} = \sum_{k=0}^{n} \left(X_{ik} Y_{kj}\right)$. (If you've been exposed to linear algebra before, this definition should be familiar to you.)

d) Suppose I give you two Boolean matrices A, B. Can you give me two matrices over real numbers, X, Y such that I can obtain AB easily from XY?